Inductance of Circuits Containing a Superconducting Plate*

V. CELLI, B. DUVALL, AND R. MANSON
Department of Physics, University of Virginia, Charlottesville, Virginia 22903
(Received 30 December 1968; in final form 13 February 1969)

Using the London theory we investigate the inductance of a system consisting of a long straight conductor of uniform cross section situated parallel to a nearby superconducting plate. Explicit formulas are derived for the particular cases of conductors having cylindrical, rectangular, and thin-film cross sections and the limitations of the theory are discussed. The conclusions indicate that measurements of the inductance of such a system can be used to determine the penetration depth of a superconductor.

I. INTRODUCTION

The London penetration depth λ in a superconductor is usually calculated from measurements on spherical or cylindrical samples because the boundary-value problems for these geometries have been solved. On the other hand, measurements in geometries involving thin films and plates appear far easier, and if the solution of the boundary-value problems were known for these geometries, they could similarly be used to determine λ. This paper derives the solution to these boundary-value problems using the London theory[1] and calculates the inductance of a straight conductor of arbitrary constant cross section paralleling a nearby flat slab of superconductor. In Sec. II we find the vector potential for a filamentary conductor and use this result in Sec. III to determine the vector potential for conductors of rectangular and cylindrical cross sections. The inductance for cylindrical, rectangular, and thin-film conductors, with appropriate limiting expressions, is calculated in Sec. IV, and Sec. V contains a discussion of the limitations imposed on the results and the approximate errors involved in using the London theory.

II. GREEN'S FUNCTION FOR THE VECTOR POTENTIAL

We consider first of all a line current of unit intensity near a flat slab of superconducting metal. Assume the superconductor to lie in the region $y<0$ with the xz plane defining its surface and let the line current lie parallel to the z axis at the distance $y=d$. In the region $y>0$ the z component of the vector potential is given by $\nabla^2 A=(4\pi/c)\delta(x)\delta(y-d)$ and can be represented as consisting of two parts: a part due to the original line current plus an image current at $y=-d$, and an additional contribution due to the finite penetration of the magnetic field into the superconductor. Thus for $y>0$ this implies

$$A=A_i+A_h, \tag{2.1}$$

where

$$A_i(x,y)=-c^{-1}\log\{[x^2+(y-d)^2]/[x^2+(y+d)^2]\} \tag{2.2}$$

and

$$\nabla^2 A_h=0. \tag{2.3}$$

In the region of the superconductor, i.e., for $y<0$, the vector potential, which we shall call A_s, is a solution of London's equation

$$\nabla^2 A_s - A_s/\lambda^2=0, \tag{2.4}$$

where λ is the penetration depth. The most general solutions of (2.3) and (2.4) are:

$$A_h(x,y)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\times A_h(k)\exp(ikx)\exp(-|k|y);\qquad y>0 \tag{2.5}$$

$$A_s(x,y)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\times A_s(k)\exp[ikx+(k^2+\lambda^{-2})^{1/2}y];\qquad y<0. \tag{2.6}$$

To completely define the solution we need only satisfy the following two matching conditions at the surface:

$$A(x,0^+)=A(x,0^-), \tag{2.7}$$

$$(\partial A/\partial y)_{y=0^+}=(\partial A/\partial y)_{y=0^-}. \tag{2.8}$$

Since $A_i(x,0)=0$ the first condition gives simply $A_h(k)=A_s(k)$ and the second condition becomes

$$\int\frac{dk}{2\pi}\exp(ikx)[(k^2+\lambda^{-2})^{1/2}+|k|]A_h(k)=\left(\frac{\partial A_i}{\partial y}\right)_{y=0}. \tag{2.9}$$

Using (2.2) to find $\partial A_i/\partial y$ at $y=0$ and inverting the Fourier integral we find

$$A_h(k)=A_s(k)=(4\pi j\lambda^2/c)[(k^2+\lambda^{-2})^{1/2}-|k|]\times\exp(-|k|d). \tag{2.10}$$

We are now in a position to compute $A_h(x,y)$ and $A_s(x,y)$ by (2.5) and (2.6). However, it is never necessary to calculate $A_s(x,y)$ since, as shown below in Sec. IV, the integral of $\mathbf{J}\cdot\mathbf{A}$ in the superconductor

* Work supported by the U.S. Air Force Office of Scientific Research.

[1] F. London, *Superfluids* (Dover Publications, Inc., New York, 1961), Vol. I.

is always canceled by the kinetic induction of the superconducting electrons.

We have thus obtained the solution for a line current, i.e. the Green's function of the boundary-value problem at hand. The nontrivial part of this Green's function is explicitly:

$$G_h(x-x' \mid y, y') = \frac{8\pi\lambda^2}{c}\int_0^\infty \frac{dk}{2\pi}\left[(k^2+\lambda^{-2})^{1/2}-k\right] \times\cos[k(x-x')]\exp[-k(y+y')]. \quad (2.11)$$

This integral can be evaluated by noting that G_h is an integral representation of Lommel functions[2]

$$G_h(x-x' \mid y, y') = \frac{2}{c}\left\{\frac{S_{1,1}(z^+)}{z^+} - \frac{S_{0,1}(z^+)}{z^+} + \frac{S_{1,1}(z^-)}{z^-} - \frac{S_{0,1}(z^-)}{z^-}\right\}, \quad (2.12)$$

where

$$z^\pm = [y+y' \pm i(x-x')]/\lambda.$$

III. VECTOR POTENTIAL FOR VARIOUS CONDUCTORS

Now the vector potential can be calculated for any arbitrary current density $J(x, y)$ parallel to the z axis by the formula

$$A_h(x, y) = \int dx' \int dy' J(x', y') G_h(x-x' \mid y, y'). \quad (3.1)$$

In the case of a current-carrying cylinder of normal metal where the current is evenly distributed over the cross section the result is quite simple. The magnetic field around a cylindrical wire is independent of the radius, and the same is true even if it is brought near the superconducting slab; hence the vector potential in this case is of the same form as (2.11).

$$A_h(x, y) = \frac{8\pi j\lambda^2}{c}\int_0^\infty \frac{dk}{2\pi} \times\left[(k^2+\lambda^{-2})^{1/2}-k\right]\cos(kx)\exp[-k(y+d)] \quad (3.2)$$

or

$$A_h(x, y) = \frac{2j}{c}\left\{\frac{S_{1,1}(y^+)}{y^+} - \frac{S_{0,1}(y^+)}{y^+} + \frac{S_{1,1}(y^-)}{y^-} - \frac{S_{0,1}(y^-)}{y^-}\right\}, \quad (3.3)$$

where j is the total current, $y^\pm = (y+d\pm ix)/\lambda$, and d is now the distance of the axis of the conductor from the superconducting slab.

If the current is evenly distributed over a rectangular conductor of width a parallel to the superconducting slab and height b, the vector potential is readily found to be

$$A_h(x, y) = \frac{32\pi j\lambda^2}{cab}\int_0^\infty \frac{dk}{2\pi k^2}\left[(k^2+\lambda^{-2})^{1/2}-k\right] \times\sinh\tfrac{1}{2}(kb)\sin\tfrac{1}{2}(ka)\cos(kx)\exp[-k(y+d)], \quad (3.4)$$

where again d is the distance of the axis of the conductor from the superconducting slab.

IV. INDUCTANCE

The inductance L of a current-carrying system is defined by $Lj^2=2W$, where W is the total energy due to the current j. For a system containing a superconductor, W is made up of two parts[3]: the magnetic energy and the kinetic energy of the superconducting electrons. The magnetic energy can be written[4] as $(1/2c)\int d^3x\mathbf{J}\cdot\mathbf{A}$, where $\mathbf{J}$ indicates both the current density in the wire and the induced current density in the superconductor. In *simply connected* superconductors the London equation gives a relationship between the supercurrent $\mathbf{J}_s$ and the vector potential:

$$\mathbf{J}_s = -(c/4\pi\lambda^2)\mathbf{A}_s. \quad (4.1)$$

On the other hand, the kinetic energy of the superconducting electrons[3] is $(2\pi\lambda^2/c^2)\int d^3x\mathbf{J}_s^2$. Adding all contributions, we have in our case $2W=(L_i+L_h)j^2$, where L_i is the inductance of the conductor and its image, and

$$L_h = (j^2c)^{-1}\int dx\int_0^\infty dy\, J(x, y)A_h(x, y). \quad (4.2)$$

We will be interested only in L_h since L_i depends only on the geometry of the system[5] and not on the penetration depth λ.

First we calculate the inductance L_h of a cylindrical conductor whose axis is a distance d from the superconducting slab. Using the fact that A_h is a harmonic function we see that the result must be independent of the radius and we get

$$L_h = \frac{8\pi\lambda^2}{c^2}\int_0^\infty \frac{dk}{2\pi}\exp(-2kd)\left[(k^2+\lambda^{-2})^{1/2}-k\right] = (2\lambda/c^2d)\{S_{1,1}(2d/\lambda)-S_{0,1}(2d/\lambda)\}. \quad (4.3)$$

A graph of L_h as a function of λ/d is given in Fig. 1. In the limiting case $\lambda \ll d$ we can set $(k^2+\lambda^{-2})^{1/2}-k \simeq 1/\lambda$ in Eq. (4.3) and it is seen that L_h increases linearly with the penetration depth.

$$L_h = 2\lambda/c^2d. \quad (4.4)$$

To find the limiting behavior in the opposite region

[2] W. Magnus and F. Oberhettinger, *Functions of Mathematical Physics* (Chelsea Publ. Co., New York, 1949), p. 42.

[3] Reference 1, p. 65.

[4] W. Panofsky and M. Phillips, *Electricity and Magnetism* (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1962), p. 172.

[5] The inductance of parallel conductors of many types can be found in F. W. Groves, *Inductance Calculations* (Dover Publications, Inc., New York, 1962).

where $\lambda \gg d$ we make the substitution $k=(y-y^{-1})/2\lambda$ in Eq. (4.3).

$$L_h=\frac{2}{c^2}\int_1^\infty \frac{dy}{y}\exp\left(\frac{-yd}{2\lambda}\right)\{1+y^{-2}\};\qquad \frac{d}{\lambda}\ll 1. \quad (4.5)$$

The right-hand side can be expressed in terms of exponential integrals[6] and the asymptotic form is

$$L_h=(2/c^2)\{-\log(d/2\lambda)-\gamma+\tfrac{1}{2}\}, \qquad (4.6)$$

where γ is Euler's constant.

It is also of interest to calculate the inductance for a conductor of rectangular geometry. From Eqs. (4.2) and (3.4):

$$L_h=\left(\frac{8\lambda d}{cab}\right)^2\int_0^\infty \frac{dt}{t^4}e^{-2t}\sin^2\frac{ta}{2d} \times\sinh^2\left(\frac{tb}{2d}\right)\left[\left(t^2+\frac{d^2}{\lambda^2}\right)^{1/2}-t\right], \qquad (4.7)$$

where the dimensionless variable of integration is defined by $t=kd$.

In the limit $\lambda\ll d$ we can again set $(t^2+d^2/\lambda^2)^{1/2}-t\simeq d/\lambda$ and obtain linear behavior with the penetration depth

$$L_h=\left(\frac{8}{cab}\right)^2\lambda d^3\int_0^\infty \frac{dt}{t^4}e^{-2t}\sin^2\frac{ta}{2d}\sinh^2\frac{tb}{2d}. \qquad (4.8)$$

Of especial interest from an experimental point of view is the case of a thin film conductor. Letting $b\rightarrow 0$ in Eq. (4.7) shows that the inductance of such a system is

$$L_h=\left(\frac{4\lambda}{ca}\right)^2\int_0^\infty \frac{dt}{t^2}e^{-2t}\sin^2\left(\frac{ta}{2d}\right)\left[\left(t^2+\frac{d^2}{\lambda^2}\right)^{1/2}-t\right] \qquad (4.9)$$

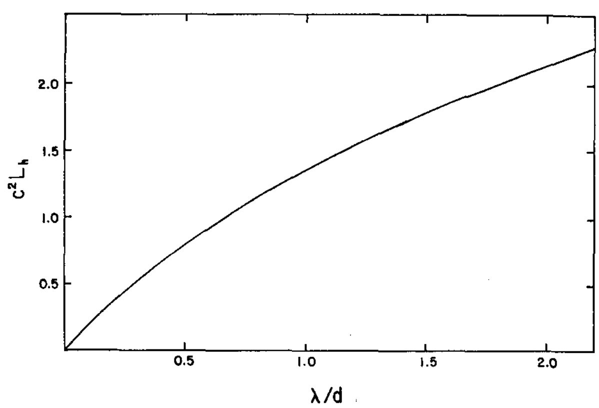

FIG. 1. The inductance per unit length L_h of a long straight cylindrical conductor whose axis parallels a flat superconducting slab at a distance d.

[6] Reference 2, p. 92.

and for $\lambda\ll d$ this shows similar linear beha ior

$$L_h=\left(\frac{4}{ca}\right)^2\lambda d\int_0^\infty \frac{dt}{t^2}e^{-2t}\sin^2\frac{ta}{2d}. \qquad (4.10)$$

However there is a much more interesting limiting behavior for a thin film with a large width. If in Eq. (4 7) we simultaneously satisfy the conditions $a\gg b$, $a\gg d$, and $a^2\gg\lambda d$ we can make use of the δ-function representation

$$\delta(t)=\lim_{g\rightarrow\infty}\sin^2(gt)/\pi g t^2 \qquad (4.11)$$

and obtain

$$L_h=4\pi\lambda/c^2a. \qquad (4.12)$$

This result may also be obtained through a simple direct calculation and can be good even for λ greater than d.

V. DISCUSSION

In the above we have derived expressions for the inductance of a linear conductor of arbitrary constant cross section situated parallel to a nearby superconducting plate. Explicit formulas were derived for conductors of cylindrical and rectangular cross sections. However, since these results were obtained using the London theory, we need to specify the limitations which the theory imposes on our results. Comparing the London theory with the more general nonlocal theory of Pippard,[7] it is seen that in our case London's Eq. (4.1) is valid when (ξ/λ) is much less than unity. Here ξ is the coherence length. Thus the above results for the inductance calculated using the London theory should be very good both for impure superconductors where ξ is small, and near the transition temperature when λ becomes large.

Finally, it should be remarked that for mathematical simplicity we have actually computed the inductance per unit length of a circuit consisting of an infinitely long conductor carrying current j and a superconducting plate carrying the return current $-j$. In practice it may be more convenient to let the conductor be part of a closed circuit with a distant return wire. The theory of Sec. III can be easily adapted to handle this system. It is seen that the dominant dependence of the inductance on λ still comes from L_h as given in Sec. IV. There are other λ-dependent contributions to the total inductance, but they are of the order λ/D, where D is the distance of the return wire from the superconducting plate.

ACKNOWLEDGMENT

We would like to thank Dr. B. Deaver for suggesting this problem to us.

[7] G. Rickayzen, *Theory of Superconductivity* (Interscience Publishers, Inc., New York, 1965), p. 48.