These are brief notes for the lecture on Friday August 20, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

1. Solving systems of linear equations

A system of linear equations is a collection of equations in the same set of variables. For example,

$$\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 - x_2 &= -4 \end{aligned}$$

Of course, since this is a pair of equations in two variables we could think of this as the intersection of two lines.

Exercise
Sketch these lines and find their intersection.

Possibilities for the solution set to a system of equations

- No solution
- Exactly one solution
- More than one solution

It is easy to come up with examples of the first two circumstances. The third possibility actually means that something much stronger is true: if we have two distinct solutions then we must have infinitely many solutions.

Exercise

Draw two dimensional examples illustrating each of the three possible outcomes.

Why is it that if we have at least two solutions then there are infinitely many?

Matrix Notation
The system of equations above has coefficient matrix

$$\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$$

and augmented matrix

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 & -4 \end{pmatrix}$$

A matrix with m rows and n columns is referred to as an $m \times n$ matrix. Note that the number of rows always comes before the number of columns. Thus the matrices above are 2×2 and 2×3 respectively.

Elementary row operations

Exercise
Subtract twice the first row from the second row of the augmented matrix. Explain why the new system of equations has precisely the same set of solutions as the old set of equations.

(1) (Replacement) Replace a row by the sum of *the same row* and a multiple of *different* row.
(2) (Interchange) Interchange two rows.
(3) (Scaling) Multiply a row by a non-zero constant.

It is clear that the set of solutions is not changed by the second and third operations.

Exercise
Explain why the set of solutions is unchanged by the first operation.

Definition 1.1. (1) *Two linear systems are* <u>*equivalent*</u> *if they have the same solution set.*
(2) *Two matrices are* <u>*row equivalent*</u> *if one matrix can be reduced to the other by elementary row operations.*

Exercise
Suppose A and B are $m \times n$ matrices: explain why the following is true: if A can be reduced via elementary row operations to B, then B can be reduced to A. (Hint: show that each step of a reduction can be reversed).

Fact: Two linear systems are equivalent if and only if their augmented matrices are row-equivalent.

Exercise
Write the following system of equations as an augmented matrix, and using that, solve the system:

$$\begin{array}{rcrcrcr} x_1 & + & & + & -3x_3 & = & 8 \\ 2x_1 & + & 2x_2 & + & 9x_3 & = & 7 \\ & + & x_2 & + & 5x_3 & = & -2 \end{array}$$

Fundamental Questions
We can rephrase our question about how many solutions there are to a system in the following way:

(1) Is the system consistent, that is, do there exist *any* solutions?
(2) If there is at least one solution, is it unique?

For example, we've seen that the system

$$\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 - x_2 &= -4 \end{aligned}$$

has a unique solution.

On the other hand, the system

$$\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 + 6x_2 &= -4 \end{aligned}$$

has no solutions, and the system

$$\begin{aligned} x_1 + 3x_2 &= 5 \\ 2x_1 + 6x_2 &= 10 \end{aligned}$$

has infinitely many.

Row Reduction and Row Echelon Form

Definition 1.2. *A rectangular matrix is in echelon form (or row echelon form) if it has the following properties*

(1) *All non-zero rows are above any rows of all zeros.*
(2) *Each leading entry of a row is in a column to the right of the leading entry of the row above it.*
(3) *All entries in a column below a leading entry are zeros.*

Sometimes we want more than echelon form: we can make all the leading entries 1 by multiplying by a constant, and we can subtract from rows above to zero out their entries in that column.

Definition 1.3. *A rectangular matrix is in reduced echelon form (or reduced row echelon form) if it is in row echelon form, and has the following additional properties*

(1) *The leading non-zero term of every non-zero row is 1*
(2) *Each leading 1 is the only non-zero entry in its column.*

Examples

$$\begin{pmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

is in row echelon form but is not reduced:

$$\begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

is in row echelon form but is not reduced:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

is in reduced row echelon form:

$$\begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 2 & 0 & 0 \end{pmatrix}$$

is not in row echelon form.

Theorem 1.4. *Each matrix is row equivalent to one and only one reduced echelon matrix.*

Pivot Positions How do we get a matrix into row echelon or reduced row echelon form? Look at the first non-empty column. By switching row 1 and another row, we can move the non-zero entry into the top row. This entry is now a *pivot position*
By multiplying by a non-zero number, we can force the pivot position to be 1. Then, subtracting multiples of this row we can force all the entries below the pivot position to be zero, as we can for all the entries above the pivot position.

Iterating this process of switching rows, scaling the pivot row to make the pivot entry 1, and subtracting multiples of the pivot row from rows below and above, gives a matrix in reduced row echelon form!
In order to obtain row echelon form, we only need to switch rows and subtract multiples below.

This process is often referred to as Gaussian elimination, or Gauss-Jordan elimination
Solving a system in reduced row echelon form Suppose we have a system of equations, we've written them as an augmented matrix, we've performed the above process, and arrived at the following reduced row echelon form matrix.

$$\begin{pmatrix} 1 & 6 & 9 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

Exercise
Write down the corresponding equations.

Exercise
Pair up the variables and the pivot columns: these are the <u>*basic variables*</u>*. The remaining variables are the* <u>*free variables.*</u>

Homework: Due: F 8/25/10.

(1) S. 1.1 #'s 1-33 odd, 34.
(2) S. 1.2 #'s 1-33 odd.
(3) S. 1.3 #'s 1-15 (odd), 19, 23, 25, 29.
(4) S. 1.4 #'s 1, 5, 9, 13, 15, 19, 23, 25, 29, 33.