These are brief notes for the lecture on Monday August 23, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

1. Row Reduction Reviewed

Recall that a matrix is in row echelon form if the leading non-zero entry of each row is a pivot, so that all the entries in the matrix below or to the left (or both) of it are zero. The row reduction algorithm works as follows. We start with the left-most non-zero column, working to the right and from the top down. At each stage, we will be working with the portion of the matrix which is below or to the right (or both) of the pivot.

(1) Find the left-most column containing a non-zero entry. This is a pivot column: the pivot position is at the top.
(2) Select a non-zero entry in the column to be the pivot. By interchanging rows if necessary, move the pivot into the pivot position.
(3) Use row replacement operations to change all the values below the pivot to zero.
(4) Move to the next row, and apply steps 1, 2, 3 to the remaining submatrix, namely the rows below and including the current row.
Once we've gone through all the rows, the matrix is in row echelon form.
(5) Scale pivots to be 1
(6) Use row replacement operations to change all the values above pivots to be zero. (There are technical reasons for doing this from the bottom pivot first).

Exercise
Row reduce

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{pmatrix} \to \left(\vphantom{\begin{matrix} 1 \\ 1 \\ 1 \end{matrix}} \right.$$

Solutions of Linear Equations
Recall that there are three possible outcomes:

(1) No solutions: for example
$$\begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$
(2) Unique solution: for example
$$\begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 \end{pmatrix}$$
(3) Infinitely many solutions: for example
$$\begin{pmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We write the solution set as

$$\begin{cases} x_1 & = 6 - 2x_2 \\ x_2 & \text{is free} \\ x_3 & = 3 \end{cases}$$

Note:

(1) this is called a *general solution.*

(2) x_2 is called a parameter or free variable.

Exercise

Find the general solution to the linear system with augmented matrix

$$\begin{pmatrix} 1 & 6 & 2 & -5 & -1 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix} \to \Bigg($$

If a linear system is consistent then the solution set contains either (i) a single element (the solution is unique, there are no free variables) or (ii) infinitely many elements (at least one free variable).

Solution Procedures (p24): given a linear system to solve

(1) write the augmented matrix

(2) perform row reduction to obtain echelon form: is the system consistent? (no: stop. yes: continue)

(3) perform row reduction to obtain reduced echelon form

(4) write system of equations corresponding to reduced echelon form

(5) Basic variables correspond to columns with pivots: free variables to columns without pivots

(6) Write basic variables in terms of free variables.

Vector Arithmetic (beginning section 1.3)

Vector: a matrix with one column

Example:

$$\begin{pmatrix} 1 \\ 2 \\ -5 \\ 9 \end{pmatrix}$$

Note: two vectors are equal precisely when they have the same number of rows and all their corresponding entries are equal.

Definition 1.1. *We define the sum of two vectors by*

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \\ \\ \qquad \vdots \qquad \\ \\ \end{pmatrix}$$

and the product of a scalar and a vector by

$$\alpha \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \\ \\ \quad \vdots \quad \\ \\ \end{pmatrix}$$

For example

$$\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix} \qquad \text{and} \qquad 3 \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

Geometry

Let $\underline{u}$ and $\underline{v}$ be given by

$$\underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \underline{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We thing geometrically of $\underline{u}$, $\underline{v}$, $\underline{2u}$ and $\underline{u} + \underline{v}$. as follows:

Parallelogram rule for vector addition: Suppose $\underline{u}$ and $\underline{v} \in \mathbb{R}^2$. Then $\underline{u}+\underline{u}$ corresponds to the fourth vertex of the parallelgram whose opposite vertex is $\underline{0}$ and whose other two vertices are $\underline{u}$ and $\underline{v}$.

Example: let $\underline{u} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$. Display $\underline{u}$, $-2/3\underline{u}$, $\underline{v}$ and $-2/3\underline{u} + \underline{v}$ on a graph.

$\mathbb{R}^n$:

In general we will consider vectors in $\mathbb{R}^n$, that is, having n real entries. $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$

The zero vector is $\underline{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ having n entries, each equal to 0.

Properties of R^n

(1) $\underline{u} + \underline{v} = \underline{v} + \underline{u}$.
(2) $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$
(3) $\underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{0}$
(4) $\underline{u} + -\underline{u} = -\underline{u} + \underline{u} = \underline{0}$ $\qquad (-\underline{u} = (-1)\underline{u})$
(5) $c(\underline{u} + \underline{v}) = c\underline{u} + c\underline{v}$
(6) $(c + d)\underline{u} = c\underline{u} + d\underline{u}$
(7) $c(d\underline{u}) = (cd)\underline{u}$
(8) $1.\underline{u} = \underline{u}$

Linear Combinations

Let p be a positive integer. Given vectors $\underline{u}_1, \underline{u}_2, \ldots \underline{u}_p$ in $\mathbb{R}^n$, and $c_1, c_2, \ldots c_p$ in $\mathbb{R}$, the vector

$$\underline{u} = c_1\underline{u}_1 + c_2\underline{u}_2 + \ldots c_p\underline{u}_p$$

is called a linear combination of the vectors $\underline{u}_1, \ldots \underline{u}_p$ with weights $c_1, \ldots c_p$.
Example:

$$2\begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + -2\begin{pmatrix} 2 \\ 3 \\ -8 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

is a linear combination of $\begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -8 \end{pmatrix}$ with coefficients $2, 3, -2$.

Geometry

Let $\underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Show all linear combinations of $\underline{u}$ and $\underline{u}$ on a graph.

Example: let $\underline{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ $\underline{u}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$ Is $\underline{b}$ a linear combination of $\underline{u}_1$ and $\underline{u}_2$

Vector Equation

A vector equation

$$x_1\underline{v}_1 + x_2\underline{v}_2 + \cdots + x_n\underline{v}_n = \underline{b}$$

has the same solution set as the system of equations whose augmented matrix is

$$\begin{pmatrix} | & | & \cdots & | & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n & \underline{b} \\ | & | & \cdots & | & | \end{pmatrix}$$

In particular, $\underline{b}$ is a linear combination of $\underline{v}_1, \ldots \underline{v}_n$ if and only if the system of linear equations is consistent.

Definition 1.2. *Suppose $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p \in \mathbb{R}^n$. We define*

$$\mathit{Span}(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_p) = \left\{c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots c_p\underline{v}_p : c_1, c_2, \ldots, c_p \in \mathbb{R}\right\}$$

That is, $\text{Span}(\underline{v}_1, \ldots, \underline{v}_p)$ is the set of all linear combinations of $\underline{v}_1, \ldots, \underline{v}_p$.

Geometry

Geometrically, the span of a single non-zero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a line through $\underline{0}$: the span of two non-zero vectors in $\mathbb{R}^3$ is either a plane through $\underline{0}$ or, if one vector is a scalar multiple of the other, a line through $\underline{0}$.

Example: let $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} -9 & -30 & 31 \end{pmatrix}$. $\text{Span}(\underline{v}_1, \underline{v}_2)$ is a plane in $\mathbb{R}^3$ Is $\underline{b}$ in that plane? (cf the application on p.36).