These are brief notes for the lecture on Wednesday August 25, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

1.4. The Matrix Equation $A\underline{x} = \underline{b}$

Definition. *If A is an $m \times n$ matrix with columns $\underline{a}_1, \underline{a}_2, /dots, \underline{a}_n$, and $\underline{x} \in \mathbb{R}^n$, then the product of A and $\underline{x}$, which we denote by $A\underline{x}$, is defined to be*

$$A\underline{x} = \begin{pmatrix} | & | & \cdots & | \\ \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \\ | & | & \cdots & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= x_1\underline{a}_1 + x_2\underline{a}_2 + \cdots + x_n\underline{a}_n$$

Note: $A\underline{x} \in \mathbb{R}^m$.
Example:

$$\begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} =$$

Theorem 3. *If A is an $m \times n$ matrix with columns $\underline{a}_1, \underline{a}_2, \ldots \underline{a}_n$, and if $\underline{b} \in \mathbb{R}^m$, then the matrix equation $A\underline{x} = \underline{b}$ has the same set of solutions as the vector equation*

$$x_1\underline{a}_1 + x_2\underline{a}_2 + \cdots + x_n\underline{a}_n = \underline{b}$$

which in turn has the same set of solutions as the linear system with augmented matrix

$$\begin{pmatrix} | & | & \cdots & | & | \\ \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n & \underline{b} \\ | & | & \cdots & | & | \end{pmatrix}$$

Proof This follows from our definition of the product of a matrix and a vector.

Theorem. *The equation $A\underline{x} = \underline{b}$ has a solution if and only if $\underline{b}$ is a linear combination of the columns of A.*

Example: For which vectors $\underline{b}$ is the equation $A\underline{x} = \underline{b}$ solvable, where

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 7 & 2 & 3 \\ -2 & 13 & -13 \end{pmatrix}?$$

Theorem 4. *Let A be an $m \times n$ matrix. The following statements are equivalent:*

(1) *For each $\underline{b} \in \mathbb{R}^m$, the equation $A\underline{x} = \underline{b}$ has a solution.*
(2) *Each $\underline{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.*
(3) *The columns of A span $\mathbb{R}^m$.*
(4) *In the row reduction process, A has a pivot position in every row*

(Note: part 4 refers to the coefficient matrix A, not the augmented matrix $[A \;\; \underline{b}]$
Proof: statements 1, 2 and 3 are equivalent by definition. So if we show that 1 and 4 are equivalent, we will be done.

Suppose that U is the reduced echelon form of A: then

$$[A \;\; \underline{b}] \sim \cdots \sim [U \;\; \underline{d}]$$

for some $\underline{d}$. If statement 4 is true, then since U has a pivot in every row, we clearly don't have a row of zeros in U with a non-zero element in $\underline{d}$. Hence the equation is solvable. Thus, if statement 4 is true, then 1 is true. Conversely, if 4 is false, U has a zero row: let $\underline{d}$ be the vector with a 1 in that row, and zeros elsewhere. Reversing the row reduction process we obtain a vector $\underline{b}$ for which $A\underline{x} = \underline{b}$ has not solution. Hence if statement 4 is false, so is statement 1.
Computing $A\underline{x}$:

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{pmatrix}$$

We see that b_i is given by $a_{i1}x_1 + a_{i2}x_2 + \dots a_{in}x_n$.
Example:

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & -1 \\ -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 11 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

Theorem 5. *If A is an $m \times n$ matrix and $\underline{u}, \underline{v}$ are vectors in $\mathbb{R}^n$ and $c \in R$ is a scalar then*

(1) $A(\underline{u} + \underline{v}) = A\underline{u} + A\underline{v}$
(2) $A(c\underline{u}) = c(A\underline{u})$

Proof:

1.5. Solution sets of linear equations

Definition. *A system of linear equations is called* homogeneous *if it can be written as* $A\underline{x} = \underline{0}$.

Note: $\underline{x} = \underline{0}$ is always solution to $A\underline{x} = \underline{0}$. It is called the *trivial solution.*
Fact: The homogeneous equation $A\underline{x} = \underline{0}$ has a non-trivial solution if and only if it has free variables.
Proof:

Example: Determine if the following homogeneous system has non-trivial solutions:

$$\begin{array}{rcrcrcl} 2x_1 & + & 3x_2 & + & x_3 & = & 0 \\ & & 5x_2 & - & x_3 & = & 0 \\ -x_1 & + & x_2 & - & x_3 & = & 0 \end{array}$$

Example (continued) Describe the solution set.

Note: The solution set of $A\underline{x} = \underline{0}$ can always be written as $\mathrm{Span}(\underline{v}_1, \ldots, \underline{v}_p)$ for some vectors $\underline{v}_1, \ldots, \underline{v}_p$.

Definition. *An equation of the form*

$$\underline{x} = s_1\underline{v}_1 + s_2\underline{v}_2 + \cdots + s_k\underline{v}_k$$

is said to be in vector parametric form*.*

Example: Describe all solutions to $A\underline{x} = \underline{b}$ where

$$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -3 & 4 \\ 6 & 1 & 8 \end{pmatrix} \qquad\qquad \underline{b} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}$$

THEOREM 6. *Suppose that the equation $A\underline{x} = \underline{b}$ has a solution $\underline{p}$. Then all solutions to the equation have the form*

$$\underline{w} = \underline{p} + \underline{v}_h$$

where $\underline{v}_h$ is a solution to the corresponding homogeneous equation $A\underline{x} = \underline{b}$.

That is, if $\underline{p}$ is any solution to $A\underline{x} = \underline{b}$, and the solution set of $A\underline{x} = \underline{0}$ is $\text{Span}(\underline{v}_1, \ldots, \underline{v}_k)$, then the solution set of $A\underline{x} = \underline{b}$ is

$$\underline{p} + \text{Span}(\underline{v}_1, \ldots, \underline{v}_k)$$

Proof:

1.6. Applications

Read this section in the book on your own.
Sample application: balance the chemical equation

$$KMnO_4 + MnSO_4 + H_2O \longrightarrow MnO_2 + K_2SO_4 + H_2SO_4$$

(that is, determine the proportions of potassium permanganate, manganese sulphate, water, manganese dioxide, potassium sulphate and sulphuric acid molecules so that the number of atoms of each element are preserved in the above chemical reaction).

1.7. Linear Independence

DEFINITION. *An indexed set of vectors* $\{\underline{v}_1, \ldots, \underline{v}_p\}$ *in* $\mathbb{R}^n$ *is* linearly independent *if the vector equation*

$$x_1\underline{v}_1 + \cdots + x_p\underline{v}_p = \underline{0}$$

has only the trivial solution ($\underline{x} = \underline{0}$).

Otherwise if there exist $c_1, \ldots, c_p \in \mathbb{R}$ *not all zero, so that*

$$c_1\underline{v}_1 + \cdots + c_p\underline{v}_p = \underline{0}$$

then the set is linearly dependent.

Example: Are the vectors $\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$, $\underline{v}_3 = \begin{pmatrix} -1 \\ 7 \\ 21 \end{pmatrix}$, linearly independent? If not, find a dependence.

Note: The columns of the matrix A are linearly independent if and only if the equation $A\underline{x} = \underline{0}$ has only the trivial solution.

Example: Are the columns of $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 2 \\ 5 & 3 & 3 \end{pmatrix}$ linearly idependent?

Note:
(1) A set containing a single vector $\underline{v}$ is linearly independent if and only if $\underline{v} \neq 0$.
(2) A set containing two vectors is linearly independent if and only if neither vector is a multiple of the other.
Proof:

THEOREM 7. *An indexed set $S = \{\underline{v}_1, \ldots, \underline{v}_p\}$ is linearly dependent if and only if one of the vectors in S is a linear combination of the others. In fact, S is linearly dependent if and only if either $\underline{v}_1 = \underline{0}$, or there is a j so that $\underline{v}_j$ is a linear combination of $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_{j-1}$.*

Proof:

Example: Let $\underline{u} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Describe $\text{Span}(\underline{u}, \underline{v})$. For this particular $\underline{u}, \underline{v}$ we have $\underline{w} \in \text{Span}(\underline{u}, \underline{v})$ if and only if $\{\underline{u}, \underline{v}, \underline{w}\}$ is linearly dependent. Explain.

THEOREM 8. *If a set contains more vectors than there are entries (that is, rows) in the vectors, then it is linearly dependent.*

Proof:

THEOREM 9. *If $\underline{0} \in S = \{\underline{v}_1, \ldots, \underline{v}_p\}$ then S is linearly dependent.*

Proof: