These are brief notes for the lecture on Friday August 27, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

1.6. Applications

Read this section in the book on your own.
Sample application: balance the chemical equation

$$KMnO_4 + MnSO_4 + H_2O \longrightarrow MnO_2 + K_2SO_4 + H_2SO_4$$

(that is, determine the proportions of potassium permanganate, manganese sulphate, water, manganese dioxide, potassium sulphate and sulphuric acid molecules so that the number of atoms of each element (Hydrogen, Manganese, Oxygen, Potassium and Sulphur) are preserved in the above chemical reaction).

1.7. Linear Independence

Definition. *An indexed set of vectors* $\{\underline{v}_1, \ldots, \underline{v}_p\}$ *in* $\mathbb{R}^n$ *is* linearly independent *if the vector equation*

$$x_1\underline{v}_1 + \cdots + x_p\underline{v}_p = \underline{0}$$

has only the trivial solution ($\underline{x} = \underline{0}$).

Otherwise if there exist $c_1, \ldots, c_p \in \mathbb{R}$ *not all zero, so that*

$$c_1\underline{v}_1 + \cdots + c_p\underline{v}_p = \underline{0}$$

then the set is linearly dependent.

Example: Are the vectors $\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$, $\underline{v}_3 = \begin{pmatrix} -1 \\ 7 \\ 21 \end{pmatrix}$, linearly independent? If not, find a dependence.

Note: The columns of the matrix A are linearly independent if and only if the equation $A\underline{x} = \underline{0}$ has only the trivial solution.

Example: Are the columns of $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 2 \\ 5 & 3 & 3 \end{pmatrix}$ linearly idependent?

Note:
(1) A set containing a single vector $\underline{v}$ is linearly independent if and only if $\underline{v} \neq 0$.
(2) A set containing two vectors is linearly independent if and only if neither vector is a multiple of the other.
Proof:

THEOREM 1. *An indexed set $S = \{\underline{v}_1, \ldots, \underline{v}_p\}$ is linearly dependent if and only if one of the vectors in S is a linear combination of the others. In fact, S is linearly dependent if and only if either $\underline{v}_1 = \underline{0}$, or there is a j so that $\underline{v}_j$ is a linear combination of $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_{j-1}$.*

Proof:

Example: Let $\underline{u} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Describe $\mathrm{Span}(\underline{u}, \underline{v})$. For this particular $\underline{u}, \underline{v}$ we have $\underline{w} \in \mathrm{Span}(\underline{u}, \underline{v})$ if and only if $\{\underline{u}, \underline{v}, \underline{w}\}$ is linearly dependent. Explain.

THEOREM 2. *If a set contains more vectors than there are entries (that is, rows) in the vectors, then it is linearly dependent.*

Proof:

THEOREM 3. *If* $\underline{0} \in S = \{\underline{v}_1, \ldots, \underline{v}_p\}$ *then* S *is linearly dependent.*

Proof:

Testing for linear dependence
To test whether vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_k$ are linearly dependent, construct a matrix A having them as columns. Perform row reduction on the matrix.
If every column contains a pivot, then the only solution to $A\underline{x} = \underline{0}$ is $\underline{x} = \underline{0}$, and hence the vectors are linearly independent, since then the only solution to

$$x_1\underline{v}_1 + x_2\underline{v}_2 + \cdots + x_k\underline{v}_k = \underline{0}$$

is $x_1 = x_2 = \cdots = x_k = 0$.
On the other hand, if there is a column without a pivot, then there are infinitely many solutions to the equation $A\underline{x} = \underline{0}$ (since there is a free variable in the general solution to the equation): pick a non-zero solution: it will correspond to a non-trivial solution to

$$x_1\underline{v}_1 + x_2\underline{v}_2 + \cdots + x_k\underline{v}_k = \underline{0}$$

and hence the vectors are linearly dependent.