These are brief notes for the lecture on Wednesday September 1, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

1.9. The Matrix of a Linear Transformation

Recall from last time: A function $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called a linear transformation if for every $\underline{u}, \underline{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

(1) $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$.

(2) $T(c\underline{u}) = cT(\underline{u})$

Example: Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a linear transformation, and that we know

$$T(\underline{e}_1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad T(\underline{e}_2) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

where $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We can give a complete description of T as follows:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = T(x\underline{e_1} + y\underline{e}2)$$

$$=$$

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Can you find a matrix A so that $T(\underline{x})$ is the same vector as $A\underline{x}$?
(Hint: $A\underline{x}$ is a linear combination of the columns of A)

Let $\underline{e}_j \in \mathbb{R}^n$ denote the vector having a 1 in the j^{th} row, and zeros elsewhere.

THEOREM 1. *Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A so that for every $\underline{x} \in \mathbb{R}^n$,*

$$T(\underline{x} = A\underline{x}.$$

In fact, A is the $m \times n$ matrix whose j^{th} column is $T(\underline{e}_j)$.
(i.e.

$$A = \begin{pmatrix} | & | & & | \\ T(\underline{e}_1) & T(\underline{e}_2) & \dots & T(\underline{e}_n) \\ | & | & & | \end{pmatrix}.$$

Proof:

Example: find the matrix representing the transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $T(\underline{x}) = 3\underline{x}$.

DEFINITION. *Recall the definition of onto, 1-1: a linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be*

- onto $\mathbb{R}^m$ *if every $\underline{b} \in \mathbb{R}^m$ is the image of at least one $\underline{x} \in \mathbb{R}^n$. (T is onto provided that $\forall\ \underline{b} \in \mathbb{R}^m,\ \exists\ \underline{x} \in \mathbb{R}^n$ so that $T(\underline{x}) = \underline{b}$.)*
- one-to-one *(or 1-1) if every $\underline{b} \in \mathbb{R}^m$ is the image of at most one $\underline{x} \in \mathbb{R}^n$. (That is, if $T(\underline{x}) = T(\underline{y})$ then $\underline{x} = \underline{y}$.)*

Example: T is the linear transformation with matrix

$$\begin{pmatrix} 1 & 2 & 5 & 6 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(1) If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, what are m and n?

(2) Is T 1-1? What do you have to check?

(3) Is T onto? What do you have to check?

THEOREM 11. *A linear transformation* $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is 1-1 if and only if* $T\underline{x} = \underline{0}$ *has only the trivial solution.*

Proof:

THEOREM 12. *Let* $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *be a linear transformation and let* A *be its matrix.*

(1) T *is onto if and only if the span of the columns of* A *is all of* $\mathbb{R}^m$*.*

(2) T *is 1-1 if and only if the columns of* A *are linearly independent.*

Proof:

As a consequence of this theorem we can check whether T is 1-1 or onto by row-reducing its matrix A: if there is a pivot in every row, then T is onto: if there is a pivot in every column, then T is 1-1.

Example: Let $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{pmatrix}$. Is T 1-1, onto?