These are brief notes for the lecture on Friday September 10, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

2.3. Matrix Inverses, continued

Recall the definition of invertible and of the inverse of a matrix

DEFINITION. *An* $n \times n$ *matrix is* invertible *if there exists a* $n \times n$ *matrix* B *so that* $AB = I_n = BA$. B *is called the* inverse *of* A *and is denoted by* A^{-1}. *A matrix which is not invertible is said to be* singular.

THEOREM 5. *If* A *is an invertible* $m \times m$ *matrix, then for every* $\underline{b} \in \mathbb{R}^n$, *the equation* $A\underline{x} = \underline{b}$ *has a unique solution, namely* $\underline{x} = A^{-1}\underline{b}$.

Proof:

THEOREM 6.

(1) *If* A *is invertible, then so is* A^{-1}, *and its inverse is given by*

$$(A^{-1})^{-1} = A$$

.

(2) *If* A *and* B *are invertible* $n \times n$ *matrices (that is, they are both invertible and they are the same size) then* AB *is invertible, and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(3) *If* A *is invertible, then so is* A^T, *and its inverse is given by*

$$(A^T)^{-1} = (A^{-1})^T.$$

Proof:

Elementary Row Operations Recall that when solving a linear system of equations, we converted it to a matrix (corresponding to the coefficients of the unknowns), augmented this matrix by a column matrix, and then performed elementary row operations on the resulting matrix. We were then able to read off from the matrix in reduced row echelon form whether there were solutions to the original system, if so, how many there were, and to write the general solution down in parametric form (as a particular solution plus the parametric form of the solution to the homogeneous problems).

What *are* these elementary row operations? Can we represent them by matrices?

Denoting rows r and s by R_r and R_s, the row operations are:

(1) Interchange rows R_r and R_s of a matrix.

(2) For a non-zero $c \in \mathbb{R}$, replace R_r by cR_r.

(3) Replace R_r (row r) by $R_r - cR_s$

Now, if A has column form $[\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n]$ and B has the appropriate size, then BA has column form $[B\underline{a}_1, B\underline{a}_2, \ldots, B\underline{a}_n]$.

Hence, if for each of the elementary row operations we can find a matrix which represents that operation on a vector, it will also represent the same operation on A.

(1) The matrix M having (i,j)-entry m_{ij} representing interchanging R_r and R_s agrees with the identity matrix except for four entries:

$$\begin{array}{rcl@{\qquad\qquad\qquad}rcl} m_{rr} & = & 0 & m_{rs} & = & 1 \\ m_{sr} & = & 1 & m_{ss} & = & 0 \end{array}$$

(2) The matrix M representing replacing row R_r by cR_r agrees with the identity matrix except for the (r,r)-entry, which is $m_{rr} = c$.

(3) The matrix M representing replacing R_r by $R_r - cR_s$ agrees with the identity matrix except for the the (r,s)-entry, which is $m_{rs} = -c$.

Exercise: for a matrix having 4 rows, write down the elementary matrices which

(1) switches R_1 and R_3

(2) replace R_2 by $3R_2$

(3) replaces R_2 by $R_2 + 7R_4$

Exercise: write down the inverse for each of the elementary matrices above.

Note: Suppose that we have a matrix A and that we perform a sequence of elementary row operations, having matrices E_1, E_2, $\dots$,E_k. Then applying just the first operation we obtain a matrix E_1A: applying the second operation, you obtain E_2E_1A, applying the third, $E_3E_2E_1A$, and so after applying all of the operations we end up with

$$B = E_kE_{k-1}\dots E_2E_1A.$$

We've seen already that an $n \times n$ matrix A is invertible if and only if every equation $A\underline{x} = \underline{b}$ has a unique solution: this is true if and only if the row reduced echelon form of A has a pivot in every row and column, which is if and only if the row reduced echelon form is I_n.

Hence

$$I_n = E_kE_{k-1}\dots E_2E_1A$$

and we see that

$$\begin{aligned} A^{-1} &= E_kE_{k-1}\dots E_2E_1 \\ &E_kE_{k-1}\dots E_2E_1I_n. \end{aligned}$$

This gives us a way to compute A^{-1} by hand: augment A by the *matrix* I_n, and row reduce the resulting $n \times 2n$ matrix. If A is invertible, then of course, the first half of the columns will row reduce to the identity. The remaining columns will give A^{-1}.

We state this as a theorem:

THEOREM 7. *An $n \times n$ matrix A is invertible if and only if $A \sim I_n$, in which case the sequence of elementary row operations which transform A to the identity also transform the identity matrix I_n to A^{-1}.*