These are brief notes for the lecture on Friday September 24, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

3.1. Determinants, continued

Recall, we defined the determinant of an $n \times n$ square matrix A in the following fashion: we row reduced A, using only row switches and row replacement operations, until it was in echelon form U (with ij entry u_{ij}). Then if the number of row switches is k, the determinant of A is

$$(-1)^k \prod_{i=1}^{n} u_{ii}.$$

We deduced from this some useful properties, and we stated some others.

One property which follows is that if A has column form $[\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n]$, then we can regard the determinant of an $n \times n$ matrix as a function of n variables, each of which is a vector in $\mathbb{R}^n$,

$$\det(A) = f(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n).$$

Then f has the property that

(1) If we switch two vectors, say $\underline{a}_i$ and $\underline{a}_j$, then we replace the determinant by its negative, so

$$f(\underline{a}_1, \ldots, \underline{a}_i, \ldots, \underline{a}_j, \underline{a}_n) = -f(\underline{a}_1, \ldots, \underline{a}_j, \ldots, \underline{a}_i, \underline{a}_n)$$

Such a function is called *alternating.* If such a function has a vector appearing as an argument in two different positions, by switching those two positions, we see that the determinant must be zero: so, repeated columns implies determinant 0.

(2) If we replace $\underline{a}_i$ by $\underline{a}_i + c\underline{a}_j$, then the determinant is unchanged. With a bit of work, this can be converted to the following: we can replace a sum of two vectors as follows;

$$f(\underline{b}_1 + \underline{c}_1, \underline{a}_2, \underline{a}_3, \ldots, \underline{a}_n) = f(\underline{b}_1, \underline{a}_2, \underline{a}_3, \ldots, \underline{a}_n) + f(\underline{c}_1, \underline{a}_2, \underline{a}_3, \ldots, \underline{a}_n).$$

Similar behaviour happens in every column. Such a function f is called *multilinear.*

(3) If we evaluate the determinant on the identity, we get 1. That is, if $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n$ are the standard basis for $\mathbb{R}^n$, then

$$f(\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n) = 1$$

Now, it turns out that there is exactly one function of n vectors from $\mathbb{R}^n$ having these three properties (which is, of course, the determinant). This is usually taken to be the "proper definition" of the determinant: the unique multilinear alternating n-form on $\mathbb{R}^n$ which is 1 on the standard basis.

You will probably not need to know this unless you are a math major!

A consequence is the following: if the first column of A has only one non-zero value, in the jth row, we can compute the determinant as follows:

Hence we can write the first column of A as a sum $a_{11}\underline{e}_1 + a_{21}\underline{e}_2 + \cdots + a_{n1}\underline{e}_n$, and then use linearity to obtain what is called the "co-factor expansion" of the determinant of A.

This is of little use computationally, but does enable us to prove a theoretically nice formula for the determinant. A permutation σ of $\{1, 2, \ldots, n\}$ is a 1-1 and onto function $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. Any such function can be viewed as switching pairs of numbers (e.g. 1,2,3,4 is replaced by 3,2,1,4 [switching 1 and 3] etc). There may be multiple ways to obtain a permutation by switching, but the parity of the number of switches (whether there is an even number or an odd number) is always the same, and is called the sign of σ. The set of permutations on $\{1, 2, \ldots, n\}$ is called S_n.

Theorem 1. *Let A be an $n \times n$ matrix, with entries a_{ij}. Then the determinant of A is given by*

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{sign(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)}$$

This means that the determinant is a sum of terms, each of which is a product of n elements, exactly one element from each row and column. The term gets a $+$ or a $-$ associated with it according to the sign of the corresponding permutation. This will have an implication for eigenvalues and eigenvectors later in the course.

The determinant of a matrix has a concrete geometric interpretation too:

Theorem 2. *Let A be an $n \times n$ matrix, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the corresponding linear transformation. The* unit cube *in $\mathbb{R}^n$ is formed by the standard basis vectors $\underline{e}_1, \ldots, \underline{e}_n$, and has volume equal to 1. The vectors $T\underline{e}_1, \ldots, T\underline{e}_n$ define a parallelepiped in $\mathbb{R}^n$, and the volume of the parallelepiped is equal to $|det(A)|$. The sign of the determinant is determined by whether the parallelepiped has been "reflected" or not.*

This interpretation is why determinants appear so often in multivariate calculus!