These are brief notes for the lecture on Monday September 27, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

## 3.1. Determinants, continued

The determinant of a matrix has a concrete geometric interpretation too:

Theorem 1. *Let $A$ be an $n \times n$ matrix, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the corresponding linear transformation. The* unit cube *in $\mathbb{R}^n$ is formed by the standard basis vectors $\underline{e}_1, \ldots, \underline{e}_n$, and has volume equal to 1. The vectors $T\underline{e}_1, \ldots, T\underline{e}_n$ define a parallelepiped in $\mathbb{R}^n$, and the volume of the parallelepiped is equal to $|det(A)|$. The sign of the determinant is determined by whether the parallelepiped has been "reflected" or not.*

This interpretation is why determinants appear so often in multivariate calculus!

## 4.1. Vector Spaces and Subspaces

In this section, we generalize the notion of a vector space from the examples we've seen ($\mathbb{R}^n$), to include a number of other examples. As a result, we'll be able to apply tools from linear algebra (notions like linear independence, spanning sets, linear transformation, determinants) to these other examples.

DEFINITION. *A* vector space $V$ *is a non-empty set of objects called "vectors", together with two operations, called addition, + and scalar multiplication, ·, which satisfy the following conditions:*

(1) *For all* $\underline{u}, \underline{v} \in V$, $\underline{u} + \underline{v} \in V$. *(Closed under addition).*

(2) *For all* $\underline{u}, \underline{v} \in V$, $\underline{u} + \underline{v} = \underline{v} + \underline{u}$ *(Addition is commutative).*

(3) *For all* $\underline{u}, \underline{v}, \underline{w} \in V$, $\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$ *(Addition is associative).*

(4) *There is an element* $\underline{0} \in V$ *so that for all* $\underline{u} \in V$, $\underline{u} + \underline{0} = \underline{0} + \underline{u} = \underline{u}$ *(There is an additive identity).*

(5) *For every* $\underline{u} \in V$ *there is an element* $-\underline{u}$ *so that* $\underline{u} + (-\underline{u}) = \underline{0}$ *(Every vector has an additive inverse).*

(6) *For every* $\underline{u} \in V$ *and* $c \in \mathbb{R}$, $c\underline{u} \in V$. *(Closed under scalar multiplication).*

(7) *For every* $\underline{u}, \underline{v} \in V$ *and* $c \in V$, $c(\underline{u} + \underline{v}) = c\underline{u} + c\underline{v}$. *(Distributivity of scalar multiplication and vector addition).*

(8) *For every* $\underline{u} \in V$ *and* $c, d \in \mathbb{R}$, $(c + d)\underline{u} = c\underline{u} + d\underline{u}$. *(Distributivity of scalar multiplication and scalar addition).*

(9) *For every* $\underline{u} \in V$ *and* $c, d \in \mathbb{R}$, $c(d\underline{u}) = (cd)\underline{u}$ *(Associativity of scalar multiplication).*

(10) *For every* $\underline{u} \in V$, $1 \cdot \underline{u} = \underline{u}$. *(The real number 1 is a scalar multiplicative identity).*

Note: there are a large number of conditions here: and to check whether a particular set $V$ is a vector space, we do have to check all of them. However, as tedious as this may sometimes be, it is usually straightforward, and the major point is the following:

> If the elements of a non-empty set $V$ can be added together, multiplied by constants, and stay in $V$, and things work nicely, then $V$ is a vector space.

Mostly, things "work nicely", except in sets constructed specially to show that you really *ought* to check all ten conditions!

There are some nice facts which follow from the definition: in particular, 0 and $\underline{0}$ work the way that we'd like them to. For every $\underline{u} \in V$ and $c \in \mathbb{R}$

(1) $0\underline{u} = \underline{0}$

(2) $c\underline{0} = \underline{0}$

(3) $-\underline{u} = (-1)\underline{u}$

(Note that $-\underline{u}$ refers to the additive inverse of the vector $\underline{u}$: this shows that we *can* choose to interpret it as $(-1)$ times the vector $\underline{u}$.

**Proof**:

We'll now see several examples of sets which are familiar from other areas of mathematics, but which can be viewed as vector spaces.

**Example** Let $n \geq 0$ be an integer. The set $\mathbb{P}_n$ of polynomials of degree at most $n$ consists of all polynomials of the form

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n$$

where the coefficients $a_0, a_1, \ldots a_n$ and the variable $t$ are real numbers. The *degree* of $p(t)$ is the highest power of $t$ whose coefficient is not zero. If $p(t) = a_0 \neq 0$, then the degree of $p(t)$ is zero. If all the coefficients of $p(t)$ are zero, then we call $p(t)$ the *zero polynomial*: its degree is technically speaking undefined, but we include it in the set $\mathbb{P}_n$ too.

We can add two polynomials:

We can multiply a polynomial by a scalar:

The set $\mathbb{P}_n$ is a vector space. The zero polynomial is the zero vector.

**Example**: Let $\mathbb{P}_n$ be the set of all polynomials, that is $\mathbb{P} = \bigcup_{n \geq 0} \mathbb{P}_n$. Then $\mathbb{P}$ is also a vector space. Note also that $\mathbb{P}_0 \subseteq \mathbb{P}_1 \subseteq \mathbb{P}_2 \ldots$ and for each $n \geq 0$, $\mathbb{P}_n \subseteq \mathbb{P}$.

**Example** Let $C(0,1)$ be the set of continuous functions defined on the interval $(0,1)$. Then we can add two continuous functions and get a continuous function: we can multiply a continuous function by a constant and we still get a continuous function: and addition and multiplication "work nicely": so $C(0,1)$ is also a vector space. Here the zero vector is the function which is zero on the interval $(0,1)$.

**Example** Let $V$ be the set of functions $f(t) \in C(0,1)$ with the property that

$$\int_0^1 f(t) \, \mathrm{d}t = 0.$$

Then the sum of two functions with integral zero is a function whose integral is zero. If we multiply $f$ by a scalar, we still get a function whose integral is zero. Addition and multiplication "work nicely", so this is also a vector space. What is the zero vector?

**Example** Let $V$ be the set of functions $f \in C(0,1)$ for which $f(1/2) = 0$. Is this a vector space?

**Example** Let $V$ be the set of functions $f \in C(0,1)$ for which $f(1/2) = 1$. Is this a vector space?