These are brief notes for the lecture on Friday October 1, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

4.1. Vector Spaces and Subspaces, continued

Subspaces of a vector space

DEFINITION. *If V is a subspace with respect to $+, \cdot$, with zero vector $\underline{0}$, then a set $H \subseteq V$ is a* subspace *of V if*

(1) $\underline{0} \in H$

(2) *For every $\underline{u}, \underline{v} \in H$, $\underline{u} + \underline{v} \in H$.*

(3) *For every $\underline{u} \in H$ and $c \in \mathbb{R}$, $c\underline{u} \in H$.*

Example: For any vector space V with zero vector $\underline{0}$, the set $\{\underline{0}\}$ is a subspace of V.

Example: If $m < n$ the $\mathbb{P}_m$ is a subspace of $\mathbb{P}_n$.

Note: $\mathbb{R}^2$ is *not* a subspace of $\mathbb{R}^3$. Indeed, $\mathbb{R}^2$ is not even a *subset* of $\mathbb{R}^3$. However, a plane through the origin in $\mathbb{R}^3$ *is* a subspace of $\mathbb{R}^3$.

Recall the definitions of linear combinations and span:

DEFINITION. *Suppose that $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k \in V$ and $c_1, c_2, \ldots c_k \in \mathbb{R}$. Then*

$$\sum_{i=1}^{k} c_i \underline{v}_i$$

is the linear combination of $\underline{v}_1, \ldots, \underline{v}_k$ with weights $c_1, \ldots c_k$.

DEFINITION. *Span$(\underline{v}_1, \ldots, \underline{v}_k)$ denotes the set of all linear combinations of $\underline{v}_1, \ldots, \underline{v}_k$.*

THEOREM 1. *If V is a vector space and if $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k \in V$, then $H =$ Span$(\underline{v}_1, \ldots, \underline{v}_k)$ is a subspace of V.*

Proof:

4.2. Null Spaces, Column Spaces and Linear Transformations

Recall the definition of the null space of a matrix:

DEFINITION. *Let A be a $m \times n$ matrix, so that the transformation $\underline{x} \mapsto A\underline{x}$ maps $\mathbb{R}^n$ to $\mathbb{R}^m$. The* null space *of A is defined to be*

$$\textit{Nul } A = \{\underline{x} : \underline{x} \in \mathbb{R}^n \text{ and } A\underline{x} = \underline{0}\}.$$

That is, it is those elements of $\mathbb{R}^n$ which are mapped to $\underline{0}$ by A.

Note that Nul A is a subset of $\mathbb{R}^n$.

THEOREM 2. *If A is an $m \times n$ matrix, then Nul A is a subspace of $\mathbb{R}^n$.*

Proof:

Example: Let $H = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : a - 2b + 5c = d \text{ and } c - a = b \right\}$. Show that H is a subspace of $\mathbb{R}^4$ by expressing this as a null space of a matrix. Find a spanning set for this H.

Example: Find a spanning set for the null space of
$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & -3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$.

How do we find a spanning set for the null space of a matrix A?

First, we express the solutions to the equation $A\underline{x} = \underline{0}$ in parametric form: recall that the way that we do this is to put A into reduced row echelon form:

$$A \sim U$$

and identify the non-pivot columns: these correspond to free variables

$$x_{i_1} = t_1, x_{i_2} = t_2, \ldots, x_{i_k} = t_k.$$

Next we write the variables $x_1, x_2, \ldots, x_n$ in terms of these free variables. (Of course, the non-pivot variables are given in the equation above, so we just need to solve for the remaining variables).

Now we separate out these according to the free variables, so we'll get something of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = t_1 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix} + t_2 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} + \ldots t_k \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{pmatrix}$$

Then the vectors that we have constructed on the right hand side span the null space! That is, every solution to $A\underline{x} = \underline{0}$ is a linear combination of the vectors.

Furthermore, the vectors we've constructed are automatically linearly independent: to see this, in row i_j corresponding to the non-pivot variable x_{i_j}, the only t which occurs is t_j. Hence if the linear combination sums to $\underline{0}$, then since it is zero in the i_j^{th} position, it must have $t_j = 0$. Hence the only linear combination giving $\underline{0}$ is the trivial combination.

Let's rephrase this; when we compute a spanning set for the null space Nul A by row reducing A, we end up with a spanning set which is automatically independent. Further, when Nul $A \neq \{\underline{0}\}$, the number of vectors we obtain in this spanning set is equal to the number of free variables.

The Column Space

DEFINITION. *Let A be an $m \times n$ matrix having column form $[\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n]$. Then the column space of A, denoted Col A is given by*

$$\textit{Col } A = \textit{Span}(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n).$$

THEOREM 3. *If A is an $m \times n$ matrix, then Col A is a subspace of $\mathbb{R}^m$.*

Proof: Let A be an $m \times n$ matrix. Note that

$$\text{Col } A = \{A\underline{x} \ : \ \underline{x} \in \mathbb{R}^n\}$$

since any linear combination of the columns of A with weights $x_1, x_2, \ldots, x_n$ is of this form. Clearly Col $A \subseteq \mathbb{R}^m$ (since the columns of A are in this space, so are all linear combinations of them).

To show that Col A is a subspace of $\mathbb{R}^m$, we have to show

(1) $\underline{0} \in \text{Col } A$.

(2) If $\underline{u}, \underline{v} \in \text{Col } A$ then $\underline{u} + \underline{v} \in \text{Col } A$.

(3) If $c \in \mathbb{R}$ and $\underline{u} \in \text{Col } A$ then $c\underline{u} \in \text{Col } A$.

(1)

(2)

(3)

Example: Find a matrix A so that $\text{Col } A = \left\{ \begin{pmatrix} 5a - b \\ 3b + 2a \\ -7a \end{pmatrix} \ : \ a, b \in \mathbb{R} \right\}$

Given a matrix A, how do we find set which spans the column space?

Note: For an $m \times n$ matrix A, $\text{Col } A = \mathbb{R}^m$
$\iff$ if for every $\underline{b} \in \mathbb{R}^m$ the equation $A\underline{x} = \underline{b}$ has a solution
$\iff$ if the linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with matrix A is onto.