These are brief notes for the lecture on Monday October 4, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

4.2. Null Spaces, Column Spaces and Linear Transformations, continued

Example: Find a spanning set for the null space of
$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & -3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$.

How do we find a spanning set for the null space of a matrix A?

First, we express the solutions to the equation $A\underline{x} = \underline{0}$ in parametric form: recall that the way that we do this is to put A into reduced row echelon form:

$$A \sim U$$

and identify the non-pivot columns: these correspond to free variables

$$x_{i_1} = t_1, x_{i_2} = t_2, \ldots, x_{i_k} = t_k.$$

Next we write the variables $x_1, x_2, \ldots, x_n$ in terms of these free variables. (Of course, the non-pivot variables are given in the equation above, so we just need to solve for the remaining variables).

Now we separate out these according to the free variables, so we'll get something of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = t_1 \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix} + t_2 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} + \ldots t_k \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{pmatrix}$$

Then the vectors that we have constructed on the right hand side span the null space! That is, every solution to $A\underline{x} = \underline{0}$ is a linear combination of the vectors.

Furthermore, the vectors we've constructed are automatically linearly independent: to see this, in row i_j corresponding to the non-pivot variable x_{i_j}, the only t which occurs is t_j. Hence if the linear combination sums to $\underline{0}$, then since it is zero in the i_j^{th} position, it must have $t_j = 0$. Hence the only linear combination giving $\underline{0}$ is the trivial combination.

Let's rephrase this; when we compute a spanning set for the null space Nul A by row reducing A, we end up with a spanning set which is automatically independent. Further, when Nul $A \neq \{\underline{0}\}$, the number of vectors we obtain in this spanning set is equal to the number of free variables.

The Column Space

DEFINITION. *Let A be an $m \times n$ matrix having column form $[\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n]$. Then the column space of A, denoted Col A is given by*

$$Col\ A = Span(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n).$$

THEOREM 1. *If A is an $m \times n$ matrix, then Col A is a subspace of $\mathbb{R}^m$.*

Proof: Let A be an $m \times n$ matrix. Note that

$$\text{Col } A = \{A\underline{x} \ : \ \underline{x} \in \mathbb{R}^n\}$$

since any linear combination of the columns of A with weights $x_1, x_2, \ldots, x_n$ is of this form. Clearly Col $A \subseteq \mathbb{R}^m$ (since the columns of A are in this space, so are all linear combinations of them).

To show that Col A is a subspace of $\mathbb{R}^m$, we have to show

(1) $\underline{0} \in$ Col A.

(2) If $\underline{u}, \underline{v} \in$ Col A then $\underline{u} + \underline{v} \in$ Col A.

(3) If $c \in \mathbb{R}$ and $\underline{u} \in$ Col A then $c\underline{u} \in$ Col A.

(1)

(2)

(3)

Example: Find a matrix A so that Col $A = \left\{ \begin{pmatrix} 5a - b \\ 3b + 2a \\ -7a \end{pmatrix} \ : \ a, b \in \mathbb{R} \right\}$

Given a matrix A, how do we find a set which spans the column space?

Note: For an $m \times n$ matrix A, Col $A = \mathbb{R}^m$
$\iff$ if for every $\underline{b} \in \mathbb{R}^m$ the equation $A\underline{x} = \underline{b}$ has a solution
$\iff$ if the linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ with matrix A is <u>onto</u>.

Example: Let $A = \begin{pmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{pmatrix}$

(1) Col $A <$ ____________ ($<$ means "is a subspace of")

(2) Nul $A <$ ____________

(3) Give a vector in Col A.

(4) Describe Nul A is vector parametric form.

$$A \sim \begin{pmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(5) Let $\underline{u} = \begin{pmatrix} 3 \\ -2 \\ -1 \\ 0 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$. Is either of $\underline{u}$ or $\underline{v}$ in either Col A or Nul A?

We now extend the definition of linear transformations to cover all vector spaces.

Definition. *Suppose that U and V are vector spaces. A transformation $T : U \longrightarrow V$ is said to be linear if*

(1) *For all $\underline{u}, \underline{v} \in U$, $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$*

(2) *For all $\underline{u} \in U$ and $c \in \mathbb{R}$, $T(c\underline{u}) = cT(\underline{u})$.*

We also extend the definition of the null space and the column space of a matrix to a general linear transformation.

Definition. *Let U and V be vector spaces, and let $T : U \longrightarrow V$ be a linear transformation. The* kernel *of T is*

$$ker(T) := \{\underline{u} \in U \ : \ T(\underline{u}) = \underline{0}\}.$$

The range *of T is*

$$range(T) = im(T) := \{T(\underline{u}) \ : \ \underline{u} \in U\}.$$

Fact: Given a linear transformation $T : U \longrightarrow V$,

(1) $\ker(T) < U$.

(2) $\mathrm{im}(T) < W$.

Proof:

4.3. Linear Independence in Vector Spaces; Bases

Just as in the case of $\mathbb{R}^n$, we can define linear dependence and linear independence of sets of vectors.

DEFINITION. *Let V be a vector space and let $\underline{v}_1.\underline{v}_2, \ldots, \underline{v}_p \in V$. If the only solution to the equation $x_1\underline{v}_1 + x_2\underline{v}_2 + x_p\underline{v}_p = \underline{0}$ is the trivial solution $x_1 = x_2 = \cdots = x_p = 0$ then the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_p$ are said to be* linearly independent.

If there is *a non-trivial solution to the equation (i.e. one for which some of the x_j's are non-zero) then the vectors are said to be* linearly dependent.

THEOREM 4. *Let V be a vector space. Suppose that $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p \in V$, $p \geq 2$, and that $\underline{v}_1 \neq \underline{0}$. Then $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_p$ are linearly dependent if and only if there is a $j \leq p$ so that $\underline{v}_j$ is a linear combination of the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_{j-1}$.*