These are brief notes for the lecture on Monday October 11, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

## 4.3. Linear Independence in Vector Spaces; Bases continued

**Bases for Nul $A$ and Col $A$**

We already have seen how to find a basis for Nul $A$: row reduce $A$ to obtain a matrix in reduced row echelon form and use this to express the null space in vector parametric form. The vectors appearing will be the basis for Nul $A$.

**Example**: Let

$$B = \begin{pmatrix} 1 & 3 & 0 & 2 & 9 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find a basis for Nul $B$.

For the same matrix $B$, find a basis for Col $B$.

**Fact**: If $A \sim B$, then the linear dependencies of the columns of $A$ are exactly the same as the linear dependencies of the columns of $B$.

**Proof** of the fact: we note that if the coefficients of a linear combination of columns of $A$ are $x_1, x_2, \ldots, x_n$ are placed in a vector $\underline{x}$, then the linear combination gives $\underline{0}$ if and only if $A\underline{x} = \underline{0}$. If we consider row reducing $A$ by an elementary operation, this corresponds to replacing $A$ by $EA$, where $E$ is an elementary matrix (and hence invertible). Thus

$$EA\underline{x} = \underline{0} \qquad \Longleftrightarrow \qquad A\underline{x} = \underline{0}$$

since if $A\underline{x} = \underline{0}$ then $EA\underline{x} = E\underline{0} = \underline{0}$. Conversely if $EA\underline{x} = \underline{0}$ then $E^{-1}\underline{0} = E^{-1}EA\underline{x} = IA\underline{x} = A\underline{x}$.

Consequently the linear dependencies of $A$ are the same as the linear dependencies of $EA$, and hence are the same as the row-reduced echelon form of $A$, and so any linearly independent set of columns of $A$ corresponds exactly to a linearly independent set of vectors of $B$.

THEOREM 6. *The pivot columns of a matrix $A$ form a basis for Col $A$.*

Alternatively, we can write this as an algorithm:
**Algorithm**: To compute a basis for the column space of a matrix $A$: row reduce the matrix, and determine which columns have pivots in. The *corresponding columns in* $A$ form a basis for the column space.

**Note**: the basis is drawn from the columns of the original matrix, not from its row-reduced form.

**Note**: A basis is

- A spanning set which is as small as possible
- A linearly independent set which is as big as possible

**Example**: which of the following sets of vectors form a basis for $\mathbb{R}^3$.

(1) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right\}$

(2) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(3) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

## 4.4. Coordinate Systems

THEOREM 8 (Unique Representation Theorem). *Let $V$ be a vector space, and let $\mathcal{B} = \{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n\}$ be a basis for $V$. Then for every $\underline{v} \in V$, there is a unique vector $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ so that*

$$\underline{v} = x_1\underline{b}_1 + x_2\underline{b}_2 + \cdots + x_n\underline{b}_n.$$

**Proof**: Since $\mathcal{B}$ is a basis, we know that there is at least one representation of this form (since $\underline{v} \in V$ and $\mathcal{B}$ spans $V$, $\underline{v}$ is a linear combination of the elements of $\mathcal{B}$).

So, we just have to show that there is only one such representation: this will follow from the fact that $\mathcal{B}$ is linearly independent.

Suppose that we also have

$$\underline{v} = y_1\underline{b}_1 + y_2\underline{b}_2 + \cdots + y_n\underline{b}_n.$$

We will show that $x_1 = y_1, x_2 = y_2, \ldots$. Subtracting the two expressions for $\underline{v}$, we obtain

$$\underline{0} = (x_1 - y_1)\underline{b}_1 + (x_2 - y_2)\underline{b}_2 + \cdots + (x_n - y_n)\underline{b}_n.$$

but since $\mathcal{B}$ is linearly independent, this implies that each of the values $x_1 - y_1$, $x_2 - y_2$, etc. must be 0. Hence $x_i = y_i$ as claimed.

**Notation**: Suppose $\mathcal{B}$ and $V$ are as above. Given a vector $\underline{v} \in V$ we write

$$[\underline{v}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

provided that

$$\underline{v} = \sum_{i=1}^{n} x_i\underline{b}_i = x_1\underline{b}_1 + x_2\underline{b}_2 + \cdots + x_n\underline{b}_n.$$

We refer to this as the co-ordinate representation of $\underline{v}$ with respect to the basis $\mathcal{B}$. This will enable us to use all the tools, all the machinery we have for $\mathbb{R}^n$ to analyze general vector spaces in terms of their bases.

**Example**: Suppose that $\underline{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underline{b}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$ is a basis for $\mathbb{R}^2$. Suppose that $\underline{x}$ has a co-ordinate representation with respect to this basis

$$[\underline{x}]_{\mathcal{B}} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}.$$

Then $\underline{x} =$

Suppose that $\underline{x}$ has a co-ordinate representation with respect to this basis

$$[\underline{x}]_{\mathcal{B}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Then $\underline{x} =$

Suppose $\underline{x} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$.

Then $[\underline{x}]_{\mathcal{B}} =$

Suppose $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Then $[\underline{x}]_{\mathcal{B}} =$