These are brief notes for the lecture on Wednesday October 13, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

4.4. Coordinate Systems

THEOREM 8 (Unique Representation Theorem). *Let V be a vector space, and let $\mathcal{B} = \{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n\}$ be a basis for V. Then for every $\underline{v} \in V$, there is a unique vector $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ so that*

$$\underline{v} = x_1\underline{b}_1 + x_2\underline{b}_2 + \cdots + x_n\underline{b}_n.$$

Proof: Since $\mathcal{B}$ is a basis, we know that there is at least one representation of this form (since $\underline{v} \in V$ and $\mathcal{B}$ spans V, $\underline{v}$ is a linear combination of the elements of $\mathcal{B}$).

So, we just have to show that there is only one such representation: this will follow from the fact that $\mathcal{B}$ is linearly independent.

Suppose that we also have

$$\underline{v} = y_1\underline{b}_1 + y_2\underline{b}_2 + \cdots + y_n\underline{b}_n.$$

We will show that $x_1 = y_1, x_2 = y_2, \ldots.$ Subtracting the two expressions for $\underline{v}$, we obtain

$$\underline{0} = (x_1 - y_1)\underline{b}_1 + (x_2 - y_2)\underline{b}_2 + \cdots + (x_n - y_n)\underline{b}_n.$$

but since $\mathcal{B}$ is linearly independent, this implies that each of the values $x_1 - y_1$, $x_2 - y_2$, etc. must be 0. Hence $x_i = y_i$ as claimed.

Notation: Suppose $\mathcal{B}$ and V are as above. Given a vector $\underline{v} \in V$ we write

$$[\underline{v}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

provided that

$$\underline{v} = \sum_{i=1}^{n} x_i\underline{b}_i = x_1\underline{b}_1 + x_2\underline{b}_2 + \cdots + x_n\underline{b}_n.$$

We refer to this as the co-ordinate representation of $\underline{v}$ with respect to the basis $\mathcal{B}$. This will enable us to use all the tools, all the machinery we have for $\mathbb{R}^n$ to analyze general vector spaces in terms of their bases.

Example: Suppose that $\underline{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underline{b}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$ is a basis for $\mathbb{R}^2$. Suppose that $\underline{x}$ has a co-ordinate representation with respect to this basis

$$[\underline{x}]_{\mathcal{B}} = \begin{pmatrix} 5 \\ -7 \end{pmatrix}.$$

Then $\underline{x} =$

Suppose that $\underline{x}$ has a co-ordinate representation with respect to this basis

$$[\underline{x}]_{\mathcal{B}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Then $\underline{x} =$

Suppose $\underline{x} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$.

Then $[\underline{x}]_{\mathcal{B}} =$

Suppose $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Then $[\underline{x}]_{\mathcal{B}} =$

Co-ordinates in $\mathbb{R}^n$

Suppose that we are given a basis $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_p\}$ for the vector space $\mathbb{R}^n$. Write $P_{\mathcal{B}} = [\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_p]$

Then, since every vector $\underline{v}$ is representable uniquely as a linear combination of the vectors in $\mathcal{B}$, we have that for every $\underline{v} \in \mathbb{R}^n$ the vector equation

$$x_1\underline{b}_1 + x_2\underline{b}_2 + \cdots + x_p\underline{b}_p = \underline{v}$$

(or the corresponding matrix equation $P_{\mathcal{B}}\underline{x} = \underline{v}$) has a unique solution.

Hence when we row-reduce B we must have a matrix with a pivot in every column (the representation is unique) and in every row (every equation has a solution). Hence B must be square, and so $p = n$. Let's take this into account and start the discussion again.

Suppose that we are given a basis $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$ for the vector space $\mathbb{R}^n$. Then the vector equation

$$x_1\underline{b}_1 + x_2\underline{b}_2 + \cdots + x_p\underline{b}_n = \underline{v}$$

corresponds to the matrix equation

$$\underline{v} = P_{\mathcal{B}}\underline{x}.$$

But the values $x_1, x_2, \ldots, x_n$ in $\underline{x}$ are precisely the co-ordinates of $\underline{v}$ with respect to the basis $\mathcal{B}$. Hence

$$\underline{v} = P_{\mathcal{B}}[\underline{v}]_{\mathcal{B}}.$$

Definition. *$P_{\mathcal{B}}$ is called the* change of co-ordinates *matrix.*

Now, since every vector is expressible uniquely, it means that the equation is solveable for every $\underline{v}$, and so $P_{\mathcal{B}}$ is invertible, and then

$$P_{\mathcal{B}}^{-1}\underline{v} = P_{\mathcal{B}}^{-1}P_{\mathcal{B}}[\underline{v}]_{\mathcal{B}} = [\underline{v}]_{\mathcal{B}}$$

that is,

$$[\underline{v}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\underline{v}.$$

So we can find the co-ordinates $[\underline{v}]_{\mathcal{B}}$ with respect to $\mathcal{B}$ of the vector $\underline{v}$ by multiplying it by $P_{\mathcal{B}}^{-1}$.

Theorem 8. *Let $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$ be a basis for a vector space V. Let $T : V \to \mathbb{R}^n$ be defined by*

$$T(\underline{v}) = [\underline{v}]_{\mathcal{B}}.$$

Then T is a one-to-one *linear transformation* onto $\mathbb{R}^n$*.*

Note: We have assumed that the number of vectors in the basis is equal to n, the dimension of $\mathbb{R}^n$.

Proof: It is a linear transformation:

It is one-to-one

It is onto

We say that V is isomorphic to $\mathbb{R}^n$ (isomorphic meaning "same shape" or "same form"), which we write as $V \simeq \mathbb{R}^n$.

COROLLARY. *For any vectors* $\underline{v}, \underline{v}_1, \ldots, \underline{v}_k \in V$,

$$[\underline{v}]_{\mathcal{B}} = \underline{0} \qquad \Longleftrightarrow \qquad \underline{v} = \underline{0}$$

and

$$[c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + \underline{v}_k]_{\mathcal{B}} = c_1[\underline{v}_1]_{\mathcal{B}} + c_2[\underline{v}_2]_{\mathcal{B}} + \cdots + c_k[\underline{v}_k]_{\mathcal{B}}.$$

Example: $\mathbb{P}_3$ has basis $\mathcal{B} = \{1, x, x^2, x^3\}$. If

$$p(x) = ax^3 + bx^2 + cx + d \in \mathbb{P}_3$$

then

$$[p(x)]_{\mathcal{B}} = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}.$$

$\mathbb{P}_3 \simeq$ ____.

Example: $\mathcal{B}' = \{1, x, x^2 - x, x^3 - 3x^2 + 2x\}$ is also a basis for $\mathbb{P}_3$. For the same $p(x)$ above, compute $[p(x)]_{\mathcal{B}'}$.