These are brief notes for the lecture on Friday October 15, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

4.4. Coordinate Systems

Suppose that we are given a basis $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ for the vector space $\mathbb{R}^n$. Then the vector equation

$$x_1\underline{b}_1 + x_2\underline{b}_2 + \cdots + x_p\underline{b}_n = \underline{v}$$

corresponds to the matrix equation

$$\underline{v} = P_{\mathcal{B}}\underline{x}.$$

But the values $x_1, x_2, \dots, x_n$ in $\underline{x}$ are precisely the co-ordinates of $\underline{v}$ with respect to the basis $\mathcal{B}$. Hence

$$\underline{v} = P_{\mathcal{B}}[\underline{v}]_{\mathcal{B}}.$$

Definition. *$P_{\mathcal{B}}$ is called the* change of co-ordinates *matrix.*

Now, since every vector is expressible uniquely, it means that the equation is solveable for every $\underline{v}$, and so $P_{\mathcal{B}}$ is invertible, and then

$$P_{\mathcal{B}}^{-1}\underline{v} = P_{\mathcal{B}}^{-1}P_{\mathcal{B}}[\underline{v}]_{\mathcal{B}} = [\underline{v}]_{\mathcal{B}}$$

that is,

$$[\underline{v}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\underline{v}.$$

So we can find the co-ordinates $[\underline{v}]_{\mathcal{B}}$ with respect to $\mathcal{B}$ of the vector $\underline{v}$ by multiplying it by $P_{\mathcal{B}}^{-1}$.

Theorem 8. *Let $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ be a basis for a vector space V. Let $T : V \to \mathbb{R}^n$ be defined by*

$$T(\underline{v}) = [\underline{v}]_{\mathcal{B}}.$$

Then T is a one-to-one *linear transformation* onto $\mathbb{R}^n$.

Note: We have assumed that the number of vectors in the basis is equal to n, the dimension of $\mathbb{R}^n$.

Proof: It is a linear transformation:

It is one-to-one

It is onto

We say that V is isomorphic to $\mathbb{R}^n$ (isomorphic meaning "same shape" or "same form"), which we write as $V \simeq \mathbb{R}^n$.

COROLLARY. *For any vectors* $\underline{v}, \underline{v}_1, \dots, \underline{v}_k \in V$,

$$[\underline{v}]_{\mathcal{B}} = \underline{0} \qquad \Longleftrightarrow \qquad \underline{v} = \underline{0}$$

and

$$[c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + \underline{v}_k]_{\mathcal{B}} = c_1[\underline{v}_1]_{\mathcal{B}} + c_2[\underline{v}_2]_{\mathcal{B}} + \cdots + c_k[\underline{v}_k]_{\mathcal{B}}.$$

Example: $\mathbb{P}_3$ has basis $\mathcal{B} = \{1, x, x^2, x^3\}$. If

$$p(x) = ax^3 + bx^2 + cx + d \in \mathbb{P}_3$$

then

$$[p(x)]_{\mathcal{B}} = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}.$$

$\mathbb{P}_3 \simeq$ ____.

Example: $\mathcal{B}' = \{1, x, x^2 - x, x^3 - 3x^2 + 2x\}$ is also a basis for $\mathbb{P}_3$. For the same $p(x)$ above, compute $[p(x)]_{\mathcal{B}'}$.

Example: Consider the set $S \subset \mathbb{P}_3$.
$S = \{p(x) = 1 + x + x^3, q(x) = 2 + x^2, r(x) = 4 + 2x + x^2 + 2x^3, s(x) = 1 + x + x^2 + x^3\}$.
Is S linearly dependent or linearly independent?

Example: Let $\underline{v}_1 = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}, \underline{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \underline{x} = \begin{pmatrix} 3 \\ 12 \\ 7 \end{pmatrix}$ Then $\{\underline{v}_1, \underline{v}_2\}$ is a basis for $H = \text{span}(\underline{v}_1, \underline{v}_2)$. Is $\underline{x} \in H$ and if so, what is $[\underline{x}]_{\mathcal{B}}$?

4.5. The Dimension of a Vector Space

THEOREM 10. *If a vector space V has a basis $\mathcal{B} = \{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n\}$ of cardinality n, then any subset of V with more than n vectors in is linearly dependent.*

Proof: Suppose that $p > n$ and $\{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_p\}$ is a set of vectors in V. Then the coordinate vectors $[\underline{u}_1]_{\mathcal{B}}, [\underline{u}_2]_{\mathcal{B}}, \ldots, [\underline{u}_p]_{\mathcal{B}}$ form a linearly dependent set of vectors in $\mathbb{R}^n$ since $p > n$ and there are p of them.

Hence we can find scalars $c_1, c_2, \ldots, c_p$, not all zero, so that

$$c_1[\underline{u}_1]_{\mathcal{B}} + c_2[\underline{u}_2]_{\mathcal{B}} + \cdots + c_p[\underline{u}_p]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{(the zero vector in } \mathbb{R}^n\text{)}$$

Since the coordinate mapping is a linear transformation,

$$[c_1\underline{u}_1 + c_2\underline{u}_2 + \cdots + c_p\underline{u}_p]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

But since the coordinate mapping is one-to-one, this means that $c_1\underline{u}_1 + c_2\underline{u}_2 + \cdots + c_p\underline{u}_p = \underline{0}$, and since not all of the c_i are zero, the vectors are linearly dependent.

Theorem 11. *If V is a vector space with a basis of size n, then every basis for V has exactly n vectors.*

Proof: Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases having n and p vectors respectively. We will show that $n = p$. First, since $\mathcal{B}_1$ is a basis, and $\mathcal{B}_2$ is linearly independent, from the previous theorem we know that $p \leq n$. Similarly, since $\mathcal{B}_2$ is a basis, and $\mathcal{B}_1$ is linearly dependent, $n \leq p$. Thus $p \leq n \leq p$ and we see that $p = n$. $\square$

Recall that if V is spanned by a finite set, then by repeatedly discarding vectors which are part of a non-trivial linear combination giving zero, we can find a basis for V. This theorem says that every basis must have the same number of vectors in it.

Definition. *If V is spanned by a finite set, then V is said to be* finite dimensional, *and the* dimension *of V, written as* $\dim V$, *is the number of vectors in a basis for V. The dimension of the zero vector space $\{\underline{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be infinite dimensional.*

Example

$$\dim \mathbb{R}^n =$$

$$\dim \mathbb{P}_n =$$

$$\dim \mathbb{P} \ =$$

Example: Find the dimension of the subspace

$$H = \left\{ \begin{pmatrix} a+4b+c+2d \\ a+2b+d \\ a+5b+c+3d \\ b+d \end{pmatrix} \ : \ a,b,c,d \in \mathbb{R} \right\}$$