These are brief notes for the lecture on Wednesday October 20, 2010: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

4.6. Rank

Suppose that we are given an $m \times n$ matrix A. Instead of regarding this as a collection of n columns (as we did when we computed the column space of A), we can regard it as a collection of m rows (each having n entries).

Now, two rows can be added together, and we get a new row with the same number of entries: a row can be multiplied by a real number, and the row of all zeros acts as $\underline{0}$. Hence the set of all rows forms a vector space.

Another viewpoint is that if we take a row of A and transpose it, we get a vector in $\mathbb{R}^m$.

In either case, we can look at the subspace spanned by the set of rows of A. In the transpose viewpoint, this is the column space of the matrix A^T. We denote the span of the rows of A by Row(A), the row space of A.

THEOREM 14. *If $A \sim B$, then Row(A)=Row(B). If B is in echelon form, then the non-zero rows of B are a basis for Row(B)= Row(A).*

Proof: Clearly the second statement follows (each non-zero row contains a pivot, which we can use to show that the rows are linearly independent). So, we need to show that if $B \sim A$, then Row(B)=Row(A). It is enough to show this for $B = EA$: then iterating this process we can get from A to any other matrix similar to it.

Exercise: Let $A = \begin{pmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{pmatrix}$. Find Row($A$), Col($A$), and Nul($A$).

Note: $A \sim \begin{pmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & -20 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

THEOREM 15 (The Rank-Nullity Theorem). *Let A be an $m \times n$ matrix. Then*

$$\dim(Row(A)) = \dim(Col(A)).$$

We call this value rank(A) and further, we have

$$rank(A) + \dim(Nul(A)) = n.$$

Proof:

THEOREM 16. *Let A be an $n \times n$ matrix. The following (extra) conditions are equivalent to A being invertible:*

(m) *The columns of A are a basis for $\mathbb{R}^n$.*

(n) *Col(A)=$\mathbb{R}^n$.*

(o) $\dim(Col(A)) = n$.

(p) *rank(A)=n.*

(q) *Nul(A)=$\{\underline{0}\}$.*

(r) $\dim(Nul(A)) = 0$.

4.7. Change of Basis

When we have a vector space V with a basis $\mathcal{B} = \{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n\}$ specified, we know that every vector $\underline{x} \in V$ can be expressed uniquely as a linear combination of $\mathcal{B}$: that is, it is represented as the coordinate vector $[\underline{x}]_{\mathcal{B}}$. Recall that the entries in the coordinate vector are the coefficients of the linear combination.

Sometimes there is more than one natural basis to use for a vector space: if we have two bases $\mathcal{B}$ and $\mathcal{C}$, then each vector $\underline{x}$ can be expressed in terms of $\mathcal{B}$ as $[\underline{x}]_{\mathcal{B}}$, and in terms of

$\mathcal{C}$ as $[\underline{x}]_{\mathcal{C}}$. Now, clearly these two representations are related, and either one determines the other: we now discuss how to transform from one to the other.

Example: Suppose that we have two bases $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$ and $\mathcal{C} = \{\underline{c}_1, \underline{c}_2\}$. Since $\mathcal{C}$ is a basis, we can express V in terms of $\mathcal{C}$: suppose that we know

$$\underline{b}_1 = 4\underline{c}_1 + 3\underline{c}_2 \qquad \text{and} \qquad \underline{b}_1 = 2\underline{c}_1 + \underline{c}_2.$$

If we are given a vector $\underline{x}$ in terms of its $\mathcal{B}$-coordinates $[\underline{x}]_{\mathcal{B}}$, we can then determine $[\underline{x}]_{\mathcal{C}}$ as follows.

Suppose $[\underline{x}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, that is,

$$\underline{x} = 3\underline{b}_1 + \underline{b}_2.$$

We can then substitute the expressions for $\underline{b}_1$ and $\underline{b}_2$ to obtain

$$\begin{aligned} \underline{x} &= 3(4\underline{c}_1 + 3\underline{c}_2) + (2\underline{c}_1 + \underline{c}_2) \\ &= 14\underline{c}_1 + 10\underline{c}_2 \end{aligned}$$

Hence we obtain $[\underline{x}]_{\mathcal{C}} = \begin{pmatrix} 14 \\ 10 \end{pmatrix}$.

Let's redo the problem from a slightly different perspective: we wish to obtain $[\underline{x}]_{\mathcal{C}}$, and we know $\underline{x} = 3\underline{b}_1 + \underline{b}_2$. Hence we know

$$[\underline{x}]_{\mathcal{C}} = [3\underline{b}_1]_{\mathcal{C}} + [\underline{b}_2]_{\mathcal{C}}$$

(since an equation about vectors remains true when we express it relative to coordinates!) We can express this in matrix form by forming a matrix whose columns are $[\underline{b}_1]_{\mathcal{C}}, [\underline{b}_1]_{\mathcal{C}}$:

$$[\underline{x}]_{\mathcal{C}} = [\ [\underline{b}_1]_{\mathcal{C}}\ [\underline{b}_2]_{\mathcal{C}}] \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

But we know the columns of the matrix are

$$[\underline{b}_1]_{\mathcal{C}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad \text{and} \qquad [\underline{b}_2]_{\mathcal{C}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

so we have

$$[\underline{x}]_{\mathcal{C}} = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ 10 \end{pmatrix}.$$

We can write this method up systematically: we obtain the following theorem.

THEOREM 17. *Let $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$ and $\mathcal{C} = \{\underline{c}_1, \ldots, \underline{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ so that*

$$[\underline{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\ [\underline{x}]_{\mathcal{B}}.$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [\ [\underline{b}_1]_{\mathcal{C}}\ [\underline{b}_2]_{\mathcal{C}}\ \cdots\ [\underline{b}_n]_{\mathcal{C}}\ ].$$

The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is called the **change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$**. Multiplication by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ converts $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates.

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are linearly independent since they are the coordinates relative to $\mathcal{C}$ of the linearly independent set $\mathcal{B}$. Hence $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is invertible. So, if we multiply both sides of the equation by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1}$ we get a formula expressing $\mathcal{B}$-coordinates in terms of $\mathcal{C}$-coordinates.

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1}[\underline{x}]_{\mathcal{C}} = [\underline{x}]_{\mathcal{B}}.$$

This means that the inverse of the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is the change of basis matrix from $\mathcal{C}$ to $\mathcal{B}$.

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$$