These are brief notes for the lecture on Monday November 9, 2009: they are not complete, but they are a guide to what I want to say today. They are not guaranteed to be correct.

5.3. Eigenvectors and Eigenvalues

5.4. The Characteristic Equation

THEOREM. *A scalar λ is an eigenvalue of an $n \times n$ matrix if and only if λ satisfies the characteristic equation*

$$\det(A - \lambda I) = 0.$$

It is true, but beyond the scope of this course, that if A is an $n \times n$ matrix then the characteristic polynomial $\det(A - \lambda I)$ has degree n. Hence an $n \times n$ matrix has at most n eigenvalues. There are lots of ways to prove this: for example, using co-factor expansions of determinants, using the fact that the determinant of a matrix can be expressed as a sum of products in which exactly one term from each column appears, or using clever algebraic methods which avoid determinants entirely.

Some matrices are such that all of their eigenvalues are distinct: however, sometimes a matrix can have repeated eigenvalues: for example, the $n \times n$ matrix I has n eigenvalues, all of which are 1, since $\det(I - \lambda I) = (1 - \lambda)^n$.

If A has characteristic equation $p(\lambda) = \det(A - \lambda I)$ and if λ_i is a multiple root of this polynomial, then we define the *multiplicity* of the eigenvalue λ_i to be the number of times $\lambda - \lambda_i$ is a root of the polynomial. (Note: sometimes there are other notions of multiplicity, and we'll need to call this the "algebraic" multiplicity of λ_i.)

Example: Find the eigenvalues and their multiplicities of

$$A = \begin{pmatrix} 6 & 5 & 0 & -5 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 7 \end{pmatrix}.$$

DEFINITION. *Two $n \times n$ matrices A and B are similar if there is an invertible matrix P so that $P^{-1}AP = B$, or equivalently $A = PBP^{-1}$.*

Note that if P is invertible, then so is $Q = P^{-1}$, so if $A = PBP^{-1}$ then $B = QAQ^{-1}$, so that A is similar to B if and only if B is similar to A.

Theorem 1. *If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomials.*

Proof: If $A = PBP^{-1}$ then

$$\begin{aligned}\det(A - \lambda I) &= \det(PBP^{-1} - \lambda I) = \det(PBP^{-1} - \lambda PIP^{-1}) \\ &= \det(P(B - \lambda I)(P^{-1}) = \det(P)\det(B - \lambda I)\det(P^{-1}) \\ &= \det(P)\det(P^{-1})\det(B - \lambda I) = \det(B - \lambda I)\end{aligned}$$

as claimed.

5.5. Diagonalization

Definition. *A square matrix D having ij^{th} element d_{ij} is* diagonal *if $d_{ij} = 0$ except for the elements d_{jj} on the diagonal: that is,*

$$\textit{if } i \neq j \textit{ then } d_{ij} = 0.$$

Often, if the eigenvalues and eigenvectors of a $n \times n$ matrix A are nice enough, we can use them to obtain a nice factorization of the form $A = PDP^{-1}$. Here we want D to be a nice matrix, for example, diagonal, and P to be invertible, so that its columns form a basis for $\mathbb{R}^n$.

A big part of the reason this is a useful thing is that we can compute powers of diagonal matrices very quickly: for example, if

$$D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

then

$$D^2 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 5^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 3^2 \end{pmatrix}$$

$$D^3 = DD^2 \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 5^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 3^2 \end{pmatrix} = \begin{pmatrix} 5^3 & 0 & 0 \\ 0 & 2^3 & 0 \\ 0 & 0 & 3^3 \end{pmatrix}$$

and in general,

$$D^k = \begin{pmatrix} 5^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{pmatrix} \quad \text{for } k \geq 1.$$

If $A = PDP^{-1}$, where D is diagonal, then we can compute A^k easily in terms of D^k:

$$A^2 = (PDP^{-1})(PDP^{-1}) = (PD)(P^{-1}P)(DP^{-1}) = (PD)I(DP^{-1}) = PD^2P^{-1},$$
$$A^3 = (PDP^{-1})(PD^2P^{-1}) = (PD)(P^{-1}P)(D^2P^{-1}) = (PD)I(D^2P^{-1}) = PD^3P^{-1}$$

and in general

$$A^k = (PDP^{-1})(PD^{k-1}P^{-1}) = (PD)(P^{-1}P)(D^{k-1}P^{-1}) = (PD)I(D^{k-1}P^{-1}) = PD^kP^{-1}.$$

Without all the derivation crowding this out, it's clearer:

$$A = PDP^{-1},$$
$$A^2 = PD^2P^{-1},$$
$$A^3 = PD^3P^{-1}$$

and in general, if $k \geq 1$,

$$A^k = PD^kP^{-1}.$$

So, if we can compute D^k easily (which we can, for a diagonal matrix D) and if $A = PDP^{-1}$, then we can compute A^k easily too.

Definition. *An $n \times n$ matrix A is said to be diagonalizable if there exists a an invertible matrix P and a diagonal matrix D (both of which must be $n \times n$) so that $A = PDP^{-1}$ (or equivalently, since P is invertible, $AP = PD$): that is, A is similar to a diagonal matrix.*

Example: Let A be the matrix

$$\begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix}.$$

Find eigenvalues and eigenvectors for A.

Place the eigenvectors in a matrix P, and the corresponding eigenvalues in a diagonal matrix D

Compute PDP^{-1}.

Compute A^k.

Some matrices turn out not to be diagonalizable: we'll see shortly that, for example,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is not diagonalizable.

Eigenvalues and eigenvectors turn out to be exactly what we need to study diagonalization of matrices. In fact, we can characterize exactly when a square matrix can be diagonalized.

THEOREM 6 (The Diagonalization Theorem). *An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors. In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case the diagonal entries of D are the eigenvalues of A, with the jj^{th} entry of D being the eigenvalue corresponding to the eigenvector which is the j^{th} column of A.*

So, this means that we can diagonalize an $n \times n$ matrix A if and only if we can find a basis for $\mathbb{R}^n$ consisting of eigenvectors of A. Before we see why this is, let's revisit the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ above.

The characteristic equation for A is $\lambda^2 = 0$, so the only eigenvalue is 0. Now, the null space of A is 1 dimensional, so we can only find one linearly independent eigenvector. Hence by the theorem, A cannot be diagonalized!

Proof of the theorem: If P is invertible, and $A = PDP^{-1}$, then $AP = PD$. If

$$P = [\underline{v}_1 \ \underline{v}_2 \ \dots \ \underline{v}_n]$$

and D has diagonal entries $\lambda_1, \dots \lambda_n$, then PD has columns

$$PD = [\lambda_1 \underline{v}_1 \ \lambda_2 \underline{v}_2 \ \dots \ \lambda_n \underline{v}_n].$$

Hence $A\underline{v}_j = \lambda_j \underline{v}_j$, and $\underline{v}_j$ is an eigenvector of A. Since P is invertible, its columns are linearly independent, and hence we have n linearly independent eigenvectors of A.

Conversely, if we have n linearly independent eigenvectors, $\underline{v}_1, \dots \underline{v}_n$ (having corresponding eigenvalues $\lambda_1, \dots \lambda_n$ respectively), we construct a matrix P having them as columns. Then

$$AP = [\lambda_1 \underline{v}_1 \ \lambda_2 \underline{v}_2 \ \dots \ \lambda_n \underline{v}_n] = PD$$

Since the n vectors in $\mathbb{R}^n$ are linearly independent, P is invertible and hence

$$A = PDP^{-1}$$

and A is diagonalizable. □

COROLLARY. *An $n \times n$ matrix A having n distinct eigenvalues is diagonalizable.*

Proof: We saw earlier that eigenvectors corresponding to distinct eigenvalues are linearly independent. Each eigenvalue has at least one corresponding eigenvector: hence we have n linearly independent eigenvectors, and A is diagonalizable.