Combinatorial Analyis	**Fall 2010**

Lecture 2: August 20

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2.1 Principles of Counting

2.1.1 Two Basic Principles

1. Addition: If A, B are finite and $A \cap B = \emptyset$, that is A, B are disjoint, then $|A \cup B| = |A| + |B|$.
   (More generally, we have $|A \cup B| = |A| + |B| - |A \cap B|$.)
   (We'll extend this idea to the principle of Inclusion-Exclusion.)
2. Multiplication: If A, B are finite sets, then $|A \times B| = |A| \cdot |B|$ where $A \times B = \{(a, b) : a \in A, b \in B\}$.

Aside:
How to talk about ordered pairs if all you are allowed is sets?
We can build $A \times B$ as the set of all elements of the form $\{a, \{a, b\}\}$.
If $a = b$, then we get $\{a, \{a\}\}$ and we know the ordered pair in question was (a, a).

2.2 Combinatorial Classes

A combinatorial class $\mathcal{A}$ is a countable or finite set, together with a map (size, weight,...).

$$w : \mathcal{A} \longrightarrow \mathbb{N} = \{0, 1, 2, ...\}$$

So, each element of $\mathcal{A}$ has a "size" or "weight" associated with it.

Later, we'll have more than one : for example, partitions of n into k parts have total weight n and k parts.

Further, we require that the inverse image of each n is finite, that is $w^{-1}(n) = \{a \in \mathcal{A} : w(a) = n\}$ is finite, i.e. only finitely many elements of weight n.

So we can define a finite set

$$\mathcal{A}_n = \{a \in \mathcal{A} : w(a) = n\}$$

and we see that $\mathcal{A} = \bigcup_n \mathcal{A}_n$.

We'll write A_n or, sometimes, a_n for $|\mathcal{A}_n|$.

We'll define the generating function got the combinatorial class $\mathcal{A}$ (with x marking weight) to be

$$\sum_{a \in \mathcal{A}} x^{w(a)} = \sum_{n \geq 0} A_n x^n = A_0 + A_1 x + A_2 x^2 + ...$$

If we call this formal power series $f(x)$, we denote by $[x^n]f(x)$ the coefficient of x^n in $f(x)$. That is, $[x^n]f(x) = A_n$.

2.2.1 Facts about coefficient extraction

1. $[x^n]$ is a linear operator.
   So, $[x^n](f(x) + g(x)) = ([x^n]f(x)) + ([x^n]g(x))$.
2. $[x^n]x^k f(x) = [x^{n-k}]f(x)$
3. $$[x^n]f(x) = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(x)}{x^{n+1}} dx$$
   if $f(x)$ is analytic in the disk of radius $R < r < 0$.
4. BEWARE : We <u>CANNOT</u> do things like
   $$([x^m]f(x)) + ([x^n]f(x)) = [x^m + x^n]f(x)$$
   IT IS NONSENSICAL!

2.3 Generating Functions:

We have a natural bijection between sequences (one-way infinite) e.g. $(f_0, f_1, f_2, ...)$ and generating function $f_0 + f_1 x + f_2 x^2 + ...$ BUT often generating functions dispose of the following problem:

What is the next term in the series (1,2,4,8,16,...)? The natural choice is 32, but there could certainly be another sequence that starts this way and does not have a 32 there.

Look at the sequence whose general term is 2^n, in this case the generating function is $\sum_{n\geq 0} 2^n x^n$, call it $f(x)$.

Then,

$$\begin{aligned} 2xf(x) &= \sum_{n\geq 0} 2^{n+1}x^{n+1} \\ &= \sum_{n\geq 1} 2^n x^n \\ &= f(x) - 1 \\ \Rightarrow f(x) &= (1-2x)^{-1} \end{aligned}$$

As formal power series, $g(x) = f(x)^{-1}$ means precisely $f(x)g(x) = 1$.

2.3.1 Manipulating formal power series

1. $\sum_{n\geq 0} f_n x^n + \sum_{n\geq 0} g_n x^n = \sum_{n\geq 0} (f_n + g_n)x^n$

2. $(\sum_{k\geq 0} f_k x^k)(\sum_{l\geq 0} g_l x^l) = \sum_{k\geq 0, l\geq 0} f_k g_l x^{k+l} = \sum_{n\geq 0}(\sum_{k+l=n} f_k g_l)x^n$

Notice that item number 2 gives you what you would expect when expanding a product of polynomials i.e. $(f_0+f_1x+f_2x^2+f_3x^3+...)(g_0+g_1x+g_2x^2+g_3x^3+...) = f_0g_0+(f_0g_1+f_1g_0)x+(f_0g_2+f_1g_1+f_2g_0)x^2+...$

Exercise TBHI: Check that $(1-2x)(1+2x+2^2x^2+2^3x^3+...) = 1$

Theorem 2.1 *Suppose $\mathcal{A}$ and $\mathcal{B}$ are combinatorial classes with $\mathcal{A}\cap\mathcal{B} = \emptyset$, respective weights w_1, w_2, and generating functions $f(x), g(x)$ then $(\mathcal{A}\cup\mathcal{B}, w_s)$ is a combinatorial class with generating fucntion $f(x)+g(x)$ where*

$$w_s|_{\mathcal{A}} = w_1$$
$$w_s|_{\mathcal{B}} = w_2$$

Theorem 2.2 *For arbitrary classes $\mathcal{A}$ and $\mathcal{B}$ define:*

$$w_p : \mathcal{A}\times\mathcal{B} \to \mathbb{N}$$
$$w_p : (a,b) \to w_1(a) + w_2(b)$$

then $(\mathcal{A}\times\mathcal{B}, w_p)$ is a combinatorial class with generating function $f(x)g(x)$

Exercise TBHI: Prove this