Combinatorial Analyis **Fall 2010**

Lecture 7: September 1

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Recall: c_n is the number of ways of parenthesizing $a_1 a_2 \cdots a_n$. We have

$$c_n = \frac{1}{4n-2}\binom{2n}{n}.$$

Thus

$$c_1 = 1$$
$$c_2 = 1$$
$$c_3 = 2$$
$$c_4 = 5$$
$$c_5 = 14$$
$$c_6 = 42$$
$$c_7 = 132$$
$$c_8 = 429$$
$$c_9 = 1430$$
$$c_{10} = 4862$$

Note: $\frac{1}{2(2n-1)}\binom{2n}{n} = \frac{(2n)!}{2(2n-1)n!n!} = \frac{(2n-2)!}{n(n-1)!(n-1)!} = \frac{1}{n}\binom{2(n-1)}{(n-1)}$ gives a slightly nicer representation: fewer factors. A useful website, The On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/njas/sequences/ Observe that

$$c_1 = 1$$
$$c_2 = 1$$
$$c_3 = 2$$
$$c_4 = 5$$
$$c_5 = 14 = 2 \times 7$$
$$c_6 = 42 = 2 \times 3 \times 7$$
$$c_7 = 132 = 2^2 \times 3 \times 11$$
$$c_8 = 429 = 3 \times 11 \times 13$$

Then,

$$\frac{c_2}{c_1} = 1 = \frac{2}{2} = \frac{2 \times 1}{2}$$
$$\frac{c_3}{c_2} = 2 = \frac{6}{3} = \frac{2 \times 3}{3}$$

$$\frac{c_4}{c_3}=\frac{5}{2}=\frac{10}{4}=\frac{2\times 5}{4}$$
$$\frac{c_5}{c_4}=\frac{2\times 7}{5}=\frac{14}{5}=\frac{2\times 7}{5}$$
$$\frac{c_6}{c_5}=3=\frac{18}{6}=\frac{2\times 9}{6}$$
$$\frac{c_7}{c_6}=\frac{2\times 11}{7}=\frac{22}{7}=\frac{2\times 11}{7}$$
$$\frac{c_8}{c_7}=\frac{13}{4}=\frac{26}{8}=\frac{2\times 13}{8}$$

Therefore, we have for $n \geqslant 2$,

$$\frac{c_n}{c_{n-1}}=\frac{2\,(2n-3)}{n}.$$

Now, $c_n=\frac{c_n}{c_{n-1}}\times\frac{c_{n-1}}{c_{n-2}}\times\frac{c_{n-2}}{c_{n-3}}\times\cdots\times\frac{c_2}{c_1}\times c_1=\frac{2^{n-1}(2n-3)(2n-5)\cdots 3\cdot 1}{n(n-1)\cdots 3\cdot 2\cdot 1}$
$=\frac{1}{n}\cdot\frac{2^{n-1}(2n-3)(2n-5)\cdots 3\cdot 1\cdot(2n-2)(2n-4)\cdots 4\cdot 2}{(n-1)!2^{n-1}(n-1)!}=\frac{1}{n}\cdot\frac{(2n-2)!}{(n-1)!(n-1)!}=\frac{1}{n}\binom{2n-2}{n-1}$.

Formal power series $\leftrightarrow$ oneway infinite sequences i.e. $\sum_{n\geq 0}a_nx^n \leftrightarrow (a_0,a_1,a_2,\cdots)$

Equip the space of formal power series (or oneway infinite sequences) with coefficients in a ring (say $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\frac{\mathbb{Z}}{2\mathbb{Z}}$, etc.) with a metric $|\quad|_u$, it is actually an ultrametric via the following.

Fix $0<\gamma<1$. If $f(x)=\sum_{n\geq 0}a_nx^n$, set $|f(x)|_u=\gamma^k$ where $k=$ least n so that $a_n\neq 0$, that is $f(x)=a_kx^t\left(1+\frac{a_{k+1}}{a_k}x+\frac{a_{k+2}}{a_k}x^2+\cdots\right)$ and $a_k\neq 0$. Define $|0|_u=0$.

This metric induces a topology on the set of power series: Let $f_n(x)$, $n\geqslant 0$, be a sequence of power series, then $f_n(x)\to 0$ as $n\to\infty$ means $|f_n(x)-0|_u\to 0$, with respect to the reals, as $n\to\infty$. $\forall\varepsilon>0$, $\exists n_0$ such that if $n>n_0$ then $|f_n(x)|_u<\varepsilon \equiv$ setting $N=\frac{log\varepsilon}{log\gamma}$, this means degree of leading term of $f_n(x)>N$.

Hence $f_n(x)\to 0$ is this topology$\Leftrightarrow$ minimum degrees of $f_n(x)\to\infty$ as $n\to\infty$.

$f_n(x)\to g(x)$ if for every $N>0$, $\exists n_0$ so that if $n>n_0$ then $f_n(x)$ and $g(x)$ agree up to terms in x^N.

Exercises

1. Show that $|\quad|_u$ is a metric.
2. Show that $|f(x)\,g(x)|_u=|f(x)|_u\,|g(x)|_u$.
3. Show that $|\quad|_u$ satisfies the ultrametric inequality:

$$|f(x)+g(x)|_u\leqslant max\left\{|f(x)|_u\,,|g(x)|_u\right\}.$$

   If $|f(x)|_u\neq|g(x)|_u$, we have equality in the previous inequality.

Example:
Let $f_n(x)=\sum_{k=0}^{n}x^k$. Then $f_n(x)\to\sum_{k\geqslant 0}x^k$. Let $f_n(x)=\sum_{k=0}^{n}a_kx^k$. Then $f_n(x)\to\sum_{k\geqslant 0}a_kx^k$. Ultrametric convergence is easy!

Exercise Prove:

$\sum_{n\geqslant 0} f_n(x)$ converges with respect to $|\ \ |_u \Leftrightarrow |f_n(x)|_u \to 0$.
And, $\prod_{n\geqslant 0} h_n(x)$ converges $\Leftrightarrow h_n(x) \to 1 \Leftrightarrow h_n(x) - 1 \to 0$.

Things can still get strange: e.g. $f_n(x) = \sum_{n\geqslant 0} n!x^n$ is a perfectly valid power series.

$h(x) = (1+x)\left(1+x^2\right)\left(1+x^4\right)\left(1+x^8\right)\cdots = \prod_{i=0}^{\infty}\left(1+x^{2^i}\right)$ converges to $1+x+x^2+x^3+\cdots$.

Exercise

Prove this by

1. uniqueness of binary representations.
2. showing $(1-x)\,h(x) = 1$ and $(1-x)\left(1+x+x^2+x^3+\cdots\right) = 1$, and hence $h(x) = 1+x+x^2+x^3+\cdots$.