Combinatorial Analysis **Fall 2010**

Lecture 14: September 17

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14.1 Partitions with all parts distinct

Here the sequence $(i_1, i_2, i_3, \ldots)$ of multiplicities of each part size has entries 0/1 and the generating function for i_j is $(1 + x^j)$. So, the generating function for partitions into distinct parts is

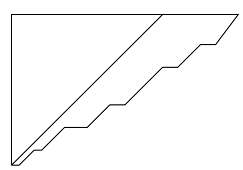
$$\prod_{j=1}^{\infty}(1 + x^j)$$

(This is very natural.)
Ferrers' diagrams for this have a nice property: (for $\lambda \vdash n$)

$$\begin{array}{lcl} \lambda_k \geq 1 & & \\ \lambda_{k-1} > \lambda_k & \text{so} & \lambda_{k-1} \geq 2 \\ \lambda_{k-2} > \lambda_{k-1} & \text{so} & \lambda_{k-2} \geq 3 \end{array}$$

Let $\mu_1, \mu_u, \ldots, \mu_k$ be given by

$$\begin{array}{rcl} \mu_1 & = & \lambda_1 - k \\ \mu_2 & = & \lambda_2 - (k-1) \\ & \vdots & \\ \mu_3 & = & \lambda_k - 1 \end{array}$$



then

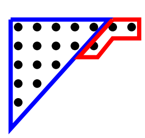
$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq 0.$$

So, discarding the zero values,

$$\mu_1 + \mu_2 + \cdots + \mu_j \vdash n - \binom{n+1}{2}.$$

Example:

$$7 + 5 + 3 + 2 + 1 \vdash 18$$



$$\begin{array}{lcccl} \mu_1 & = & 7-5 & = & 2 \\ \mu_2 & = & 5-4 & = & 1 \\ \mu_3 & = & 3-3 & = & 0 \\ \mu_4 & = & 2-2 & = & 0 \\ \mu_5 & = & 1-1 & = & 0 \end{array}$$

$$\mu = 2 + 1 \vdash 3 = 18 - \binom{6}{2}$$

μ is an arbitrary partition into at most k parts. The generating function is the same as the generating function for partitions into parts of size at most k.

Corollary 14.1

$$\prod_{j=1}^{\infty}(1+x^j) = \sum_{k=0}^{\infty} x^{\binom{k+1}{2}} \prod_{j=1}^{k}(1-x^j)^{-1}.$$

14.1.1 Remarkable Fact

$$\begin{aligned}\prod_{j=1}^{\infty}(1-x^j) &= \text{generating function for the number of partitions of n into even number of distinct parts} \\ &\quad -\text{number of partitions of n into an odd number of distinct parts} \\ &= \sum_{k=-\infty}^{\infty}(-1)^k x^{\frac{k(3k+1)}{2}}\end{aligned}$$

k	$\frac{k(3k+1)}{2}$	$(-1)^k$
-4	22	1
-3	12	-1
-2	5	1
-1	1	-1
0	0	1
1	2	-1
2	7	1
3	15	-1
4	26	1

$\dfrac{k(3k+1)}{2}$ are the "pentagonal numbers"

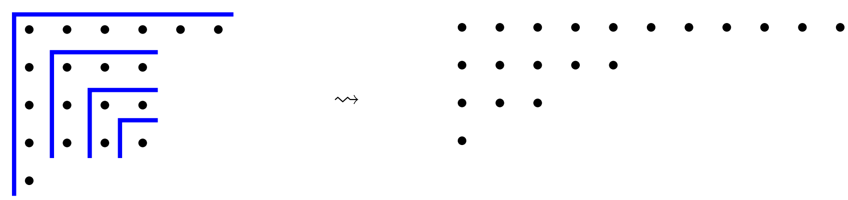
So,

$$\sum_{k=-\infty}^{\infty}(-1)^k x^{\frac{k(3k+1)}{2}} = 1 - x - x^2 + x^5 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots$$

We'll see a proof rather later using the Jordan Triple Product.
Exercise: Give a near bijection between partitions into an odd number of distinct parts and partitions into an even number of distinct parts. It will be a bijection when n is not of the form $\frac{k(3k+1)}{2}$, and off by 1 when n is of the form $\frac{k(3k+1)}{2}$. (You might need to perform surgery on the Ferrers' diagram.)

14.2 Self-conjugate partitions

$\lambda = \lambda'$. (What is the generating function for these?)
Example:



The picture on the right is a partition with distinct odd parts. The generating function for this is

$$\prod_{j=0}^{\infty}(1+x^{2j+1}).$$

<u>Exercise:</u> How many partitions are there which are self conjugate and all parts are distinct?

14.3 Back to partitions with all parts distinct

$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\cdots$

$$\begin{aligned}
&= \frac{(1-x)}{(1-x)}(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\cdots \\
&= \frac{(1-x^2)}{(1-x)}(1+x^2)(1+x^3)(1+x^4)(1+x^5)\cdots \\
&= \frac{(1-x^4)}{(1-x)}(1+x^3)(1+x^4)(1+x^5)\cdots \\
&= \frac{(1-x^3)}{(1-x)(1-x^3)}(1+x^3)(1-x^4)(1+x^4)(1+x^5)\cdots \\
&= \frac{1}{(1-x)(1-x^3)}(1-x^6)(1-x^8)(1+x^5)(1+x^6)(1+x^7)(1+x^8)\cdots \\
&= \frac{1}{(1-x)(1-x^3)(1-x^5)}(1-x^6)(1+x^6)(1+x^7)(1-x^8)(1+x^8)(1+x^9)(1-x^{10})(1+x^{10})\cdots \\
&= \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)(1-x^9)\cdots} \\
&= \prod_{j=0}^{\infty}(1-x^{2j+1})^{-1}
\end{aligned}$$

So the number of partitions of n into distinct parts is the same as the number of partitions of n into odd parts.

7	7
$6+1$	$5+1+1$
$5+2$	$3+3+1$
$4+3$	$3+1+1+1+1$
$4+2+1$	$1+1+1+1+1+1+1$

Exercise: Give a bijection.
Fact:

$$\begin{aligned} p_{5n+4} &\equiv 0 \mod 5 \\ p_{7n+5} &\equiv 0 \mod 7 \\ p_{11n+6} &\equiv 0 \mod 11 \end{aligned}$$

(These do not generalize! This fact was conjectured by the extremely talented, late, Indian mathematician, Srinivasa Ramanujan, then proven much later. These results are important because they are fascinating! And vice-versa!)