Combinatorial Analysis	**Fall 2010**

Lecture 14: September 20

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14.1 Finding upper bound for P_n

$$\sum_{n\geq 0} P_n x^n = \prod_{j=1}^{\infty}(1-x^j)^{-1}$$

$$P_n \leq \frac{\prod_{j=1}^{\infty}(1-x^j)^{-1}}{x^n} \qquad \forall x \in (0,1)$$

Let x_n^* be the minimizing value for $P(x)/x^n$ We'll rewrite $P(x)$ as

$$\begin{aligned}
P(x) &= \exp\left\{-\sum_{j=1}^{\infty}\log(1-x^j)\right\} \\
&= \exp\left\{\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{x^{jk}}{k}\right\} \\
&= \exp\left\{\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=1}^{\infty}x^{jk}\right\} \\
&= \exp\left\{\sum_{k=1}^{\infty}\frac{1}{k}\frac{x^k}{1-x^k}\right\} \\
&= \exp\left\{\sum_{k=1}^{\infty}\frac{1}{1-x}\frac{(1-x)x^k}{k(1-x^k)}\right\}
\end{aligned}$$

Aside:

$$\frac{1-x}{1-x^k}x^k < \frac{1}{k} \qquad \forall x < 1$$

$$\text{and } \lim_{x\to 1}\frac{(1-x)x^k}{1-x^k} = \frac{1}{k}$$

Exercise : Prove this Aside

Thus

$$
\begin{aligned}
P(x) &< \exp\left\{\sum_{k=1}^{\infty} \frac{1}{1-x}\frac{(1-x)x^k}{k(1-x^k)}\right\} \\
&= \exp\left\{\frac{\pi^2}{6(1-x)}\right\} \\
&\exp\left\{\frac{\pi^2}{6\epsilon} log(1-\epsilon)\right\} \\
\text{So } P_n &< \exp\left\{\frac{\pi^2}{6(1-x)}\right\}
\end{aligned}
$$

Minimize the exponent:

$$
\begin{aligned}
-\pi^2 6\epsilon^2 + \frac{n}{1-\epsilon} &= 0 \\
\text{Or: } 6\epsilon^2 n + \epsilon\pi^2 - \pi^2 &= 0 \\
\epsilon &= \frac{-\pi^2 + -\sqrt{\pi^4 + 4(6)(\pi^2)(n)}}{12n} \\
&= \frac{-\pi^2}{12n} + \sqrt{\frac{\pi^2}{6n} + \frac{\pi^4}{144n}} \\
&\simeq \sqrt{\frac{\pi^2}{6n}} \text{ (for now)} \\
\text{Hence: } P_n &< \frac{\exp\left\{\frac{\pi^2}{6}\frac{1}{\sqrt{\frac{\pi^2}{6n}}}\right\}}{(1-\frac{1}{\sqrt{\frac{\pi^2}{6n}}})^n} \\
(1-\sqrt{\frac{\pi^2}{6n}})^n &\simeq \exp\left\{\pi\sqrt{\frac{2n}{3}}\right\}(1+O(\frac{1}{n})) \\
P_n &< \exp\left\{2\sqrt{\frac{\pi^2 n}{6}}\right\} \\
&= \exp\left\{\pi\sqrt{\frac{2n}{3}}\right\}(1+O(\frac{1}{n}))
\end{aligned}
$$

This is actually a very good upper bound for P_n. In particular, the number of digits of P_n is $O(\sqrt{n})$.

$$
\begin{aligned}
\epsilon &= \frac{\sqrt{\pi^4 + 24(\pi^2)(n)}}{12n} \\
&= \sqrt{\frac{\pi^2 n}{6}}\left(\frac{\pi^2}{24n}\right)^{\frac{1}{2}} \\
&= \sqrt{\frac{\pi^2 n}{6}}\left(1 + \frac{1}{2}\left(\frac{\pi^2}{24n}\right) - \frac{1}{2}\frac{1}{2}\frac{1}{2}\left(\frac{\pi^2}{24n}\right)^2 \dots\right)
\end{aligned}
$$

So the minimizing point is actually

$$\sqrt{\frac{\pi^2}{6n}} - \frac{\pi^2}{12n} + O(\frac{1}{n^{\frac{3}{2}}}) = \sqrt{\frac{\pi^2}{6n}}(1 - \frac{1}{2}\sqrt{\frac{\pi^2}{6n}} + O(\frac{1}{n})),$$

which can be written as

$$\epsilon(1 - \frac{\epsilon}{2} + O(\frac{1}{n})).$$

Since

$$\begin{aligned} \frac{\pi^2}{6\epsilon(1-\frac{\epsilon}{2}+O(\frac{1}{n}))} &= \frac{\pi^2}{6\epsilon}(1 + \frac{\epsilon}{2} - O(\frac{1}{n})) \\ &= \frac{\pi^2}{6\epsilon} + \frac{\pi^2}{12} + O(\frac{1}{n}), \end{aligned}$$

the numerator becomes

$$\exp\{\frac{\pi^2}{6\epsilon}\}\exp\{\frac{\pi^2}{12}\},$$

and the denominator becomes

$$\begin{aligned} \frac{1}{(1-\epsilon(1-\frac{\epsilon}{2}+O(\frac{1}{n}))^n} &= \frac{1}{(1-\epsilon+\frac{\epsilon^2}{2}+O(\frac{\epsilon}{n}))^n} \\ &= \frac{1}{((1-\epsilon)(1+\frac{\epsilon^2}{2})(1+O(\frac{1}{n^{\frac{3}{2}}})))^n} \\ &= \frac{1}{(1-\epsilon)^n(1+\frac{\pi^2}{6n})^n(1+O(1))} \\ &\approx e^{\frac{\pi^2}{6}}. \end{aligned}$$

So our upper bound takes the form of $e^{\pi\sqrt{\frac{2n}{3}}}e^{-\frac{\pi^2}{12}}$.
So our bound just changes slightly.
Truth: $P_n \approx \frac{c}{n}e^{\pi\sqrt{\frac{2n}{3}}}$.
<u>Remark:</u> We could try to be more precise, and estimate how big $\frac{1}{k} - \frac{x^k}{1-x^k}(1-x)$ is, and get a better upper bound, but it isn't worth the effort.
Reason is: If ϵ^k is big, then $(1-x)^k \approx e^{-\epsilon k}$ is small. $k >> \sqrt{n}$, so the part of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ which we lose is less than

$$\sum_{k \geq \sqrt{n}} \frac{1}{k^2} \approx \int_{\sqrt{n}}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{\sqrt{n}}.$$

So $P(x)$ looks like

$$\exp\{\frac{1}{\epsilon}(\frac{\pi^2}{6} - \frac{c}{\sqrt{n}})\} = \exp\{\frac{\pi^2}{6\epsilon} - \frac{c'}{n}\} = \exp\{\frac{\pi^2}{6\epsilon}\}(1 + O(1)).$$