THE AUSTRALIAN NATIONAL UNIVERSITY
DEPARTMENT OF STATISTICS AND ECONOMETRICS

STATISTICAL INFERENCE - STAT3013/STAT8027

Mid-Semester Examination 2000 - Solutions

Total Marks: 50

Reading Period: 15 Minutes
Time Allowed: Two Hours
Permitted Materials: Course Brick, Lecture Notes, Non-Programmable Calculator

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Question 1

(a) TRUE

(b) FALSE

(c) FALSE

(d) TRUE

(e) FALSE

Question 2

(a) We can write the density function as:

$$f_X(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} \exp\left\{ -\frac{1}{2\theta}(x-\theta)^2 \right\} = \exp\left\{ -\frac{1}{2\theta}(x^2 - 2\theta x + \theta^2) - \frac{1}{2}\ln(2\pi\theta) \right\}$$
$$= \exp\left\{ -\frac{1}{2\theta}x^2 + x - \frac{1}{2}\theta - \frac{1}{2}\ln(2\pi\theta) \right\},$$

which has the form of a one-parameter exponential family with $d_1(x) = x^2$. Therefore, we know that $D = \sum_{i=1}^n X_i^2$ is a minimal sufficient statistic for θ based on a sample of size n, $X_1, \ldots, X_n$.

(b) From part (a), we see that

$$l(\theta) = \sum_{i=1}^n \ln\{f_X(X_i;\theta)\} = -\frac{1}{2\theta}\sum_{i=1}^n X_i^2 + \sum_{i=1}^n X_i - \frac{n}{2}\theta - \frac{n}{2}\ln(2\pi\theta).$$

Thus, we have:

$$l'(\theta) = \frac{1}{2\theta^2}\sum_{i=1}^n X_i^2 - \frac{n}{2} - \frac{n}{2\theta},$$

and

$$l''(\theta) = -\frac{1}{\theta^3}\sum_{i=1}^n X_i^2 + \frac{n}{2\theta^2}.$$

Therefore, the expected Fisher information is:

$$I(\theta) = -E\{l''(\theta)\} = \frac{1}{\theta^3}\sum_{i=1}^{n} E(X_i^2) - \frac{n}{2\theta^2} = \frac{n}{\theta^3}(\theta^2+\theta) - \frac{n}{2\theta^2} = \frac{n}{\theta} + \frac{n}{2\theta^2},$$

where we have used the fact that $E(X_i^2) = Var(X_i) + \{E(X_i)\}^2 = \theta + \theta^2$. Finally, then, the Cramér-Rao bound for the variance of unbiased estimators of θ is given by:

$$\left(\frac{n}{\theta} + \frac{n}{2\theta^2}\right)^{-1} = \frac{2\theta^2}{n(2\theta+1)}.$$

Now, $Var(\overline{X}) = \frac{1}{n}Var(X_1) = \frac{\theta}{n}$ which is clearly larger than $\left(\frac{n}{\theta} + \frac{n}{2\theta^2}\right)^{-1}$ for all $\theta > 0$ [since clearly $\frac{n}{\theta} < \frac{n}{\theta} + \frac{n}{2\theta^2}$].

(c) Since $\overline{X}$ is not a function of $\sum_{i=1}^n X_i^2$ which is a minimal sufficient, complete statistic (since we are dealing with a full-rank exponential family here), it cannot be the *UMVU* estimator. Indeed, to find the *UMVU* estimator in this case, we simply need to calculate $E\left(\overline{X}\middle|\sum_{i=1}^n X_i^2\right)$ (since $\overline{X}$ is clearly an unbiased estimator).

Question 3

(a) We can write the likelihood function as:

$$L(\theta) = \prod_{i=1}^{n} f_X(X_i;\theta) = \theta^n \prod_{i=1}^{n} X_i^{\theta-1} = \theta^n e^{(\theta-1)\ln(\prod_{i=1}^n X_i)}.$$

Therefore, the posterior distribution has the form:

$$\begin{aligned}\pi(\theta|X_1,\ldots,X_n) &= \frac{L(\theta)\pi(\theta)}{\int_\Theta L(t)\pi(t)dt} = c_1\theta^n e^{(\theta-1)\ln(\prod_{i=1}^n X_i)}\theta^{\alpha-1}e^{-\alpha\theta} \\ &= c_2\theta^{n+\alpha-1}e^{-\{\alpha-\sum_{i=1}^n \ln(X_i)\}\theta},\end{aligned}$$

where $c_1 = \frac{\alpha^\alpha}{\Gamma(\alpha)\int_\Theta L(t)\pi(t)dt}$ and $c_2 = c_1 e^{-\sum_{i=1}^n \ln(X_i)}$. Clearly, this has the form of a Gamma distribution with shape parameter $\alpha + n$ and scale parameter $\left\{\alpha - \sum_{i=1}^n \ln(X_i)\right\}^{-1}$.

(b) The posterior Bayes estimator is just the mean of the posterior distribution. Therefore, $\hat{\theta}_\pi = \frac{\alpha+n}{\alpha-\ln(\prod_{i=1}^n X_i)}$.

(c) Using the *MLE* we can write the posterior Bayes estimator as:

$$\hat{\theta}_\pi = \frac{\alpha+n}{\alpha - \ln(\prod_{i=1}^n X_i)} = \frac{\alpha+n}{\alpha + (n/\hat{\theta}_{MLE})} = \frac{\alpha\hat{\theta}_{MLE} + n\hat{\theta}_{MLE}}{\alpha\hat{\theta}_{MLE} + n}.$$

So, for a fixed α, we see that as n tends to infinity the posterior Bayes estimator tends towards $\hat{\theta}_{MLE}$ as it should. Similarly, for a fixed n, we see that as α tends to infinity (which corresponds to the variance of the posterior tending to zero), we see that the posterior Bayes estimator tends towards 1 (which is the mean of the posterior distribution).

Question 4

(a) When we remove X_1, we can see that the average of the remaining first components is $\frac{1}{2}(3+4) = 3.5$, the average of the remaining second components is $\frac{1}{2}(2+6) = 4$ and the average of the remaining ratios is $\frac{1}{2}\left(\frac{3}{2}+\frac{4}{6}\right) = 1.083$. Similarly, when we remove X_2, the corresponding values are 2.5, 5 and 0.458. For removal of X_3 we get 2, 3 and 0.875. Therefore, for T_1, we have $\hat{\theta}_1 = \frac{3.5}{4} = 0.875$, $\hat{\theta}_2 = \frac{2.5}{5} = 0.5$ and $\hat{\theta}_3 = \frac{2}{3} = 0.667$ and the average of these three values is $\hat{\theta}_\bullet = 0.681$. Also, T_1 itself is equal to 0.667. Thus, the Jackknife estimate of bias is $\hat{B}_J = (3-1)(0.681 - 0.667) = 0.028$. For T_2, we see that $\hat{\theta}_1 = 1.083$, $\hat{\theta}_2 = 0.458$, $\hat{\theta}_3 = 0.875$ and this means that $\hat{\theta}_\bullet = 0.806$. Finally, then, we see that T_2 is 0.806 which means that the Jackknife estimate of variance is zero.

(b) Using the alternate formula derived in Tutorial 6, we see that the Jackknife estimate of variance is calculated as:

$$\frac{n-1}{n}\sum_{i=1}^{n}(\hat{\theta}_i - \hat{\theta})^2.$$

Therefore, for T_1 we have a Jackknife variance estimate of 0.047. Similarly, for T_2 we have a Jackknife variance estimate of 0.135.

(c) We can approximate the *MSE* of these two estimators as $MSE_{t_1} = 0.047 + (0.028)^2 = 0.04784$ and $MSE_{t_2} = 0.135$. As such, we might prefer T_1. Alternatively, it appears that T_2 is unbiased in this case, so we might prefer it on those grounds. Of course, we should note that these *MSE* estimates are not necessarily that reliable. In particular, since T_2 is just an average, it is easily seen that the Jackknife bias estimate must be zero, even though it is rarely the case that T_2 is actually unbiased.

END OF EXAMINATION