Exercise H8.1. Let $X_1, \ldots, X_n$ be independent identically distributed with unknown distribution Q, where it is known only that

$$\mathrm{Var}(X_1) \leq K$$

for some known positive K. Then also $\mu = EX_1$ exists. Consider hypotheses
$H : \mu = \mu_0$
$K : \mu \neq \mu_0$.
Find an α-test (exact level, not just asymptotic). (**Hint:** Chebyshev).

Solution: from Chebyshev inequality, we have

$$\begin{aligned} \Pr\left(|\bar{X_n} - \mu_0| \geq c\right) &\leq \frac{Var\left(\bar{X_n}\right)}{c^2} \\ &= \frac{K}{nc^2}. \end{aligned}$$

Let $\alpha = \frac{K}{nc^2}$, i.e., $c = \left(\frac{K}{n\alpha}\right)^{1/2}$, and define

$$\Phi\left(X\right) = 1_A\left(|\bar{X_n} - \mu_0|\right),\ A = \left\{t : t \geq \left(\frac{K}{n\alpha}\right)^{1/2}\right\},$$

then $\Phi\left(X\right)$ is an $\alpha-$test.

Exercise H8.2. Let $X_1, \ldots, X_n$, be independent Poisson Po(λ). Consider some λ_0, λ_1 such that $0 < \lambda_0 < \lambda_1$.

(i) Consider simple hypotheses
$H : \lambda = \lambda_0$
$K : \lambda = \lambda_1$.
Find a most powerful α-test.

Note: the distribution of any proposed test statistic can be expected to be discrete, so that a *randomized* test might be most powerful. For the solution, this aspect can be ignored; just indicate the statistic, its distribution under H and the type of rejection region (such as "reject when T is too large").

(ii) Consider composite hypotheses
$H : \lambda = \lambda_0$
$K : \lambda > \lambda_0$.
Find a uniformly most powerful (UMP) α-test.

Hint: take a solution of (i) which does not depend on λ_1.

(iii) Consider composite hypotheses
$H : \lambda \le \lambda_0$
$K : \lambda > \lambda_1$.
Find a uniformly most powerful (UMP) α-test.

Hint: take a solution of (ii) and show that it preserves level α on $H : \lambda \le \lambda_0$. Properties of the Poisson distribution are useful.

Solution: (**i**) (**ii**) we have

$$\begin{aligned} L(x) &= \frac{\prod_{i=1}^{n}(e^{-\lambda_1}\lambda_1^{x_i}/x_i!)}{\prod_{i=1}^{n}(e^{-\lambda_0}\lambda_0^{x_i}/x_i!)} \\ &= e^{-n(\lambda_1-\lambda_0)}(\lambda_1/\lambda_0)^{\sum_{i=1}^{n} x_i}, \end{aligned}$$

which is an strictly increasing function of $\sum_{i=1}^{n} x_i$, for $\lambda_1 > \lambda_0$.

This implies $L(x) > t_L$, for some constant t_L , is equivalent to $\sum_{i=1}^{n} x_i > t_M$, for some corresponding constant t_M.

Thus to find an UMP randomized Neyman-Pearson test of level α is equivalent to find a test

$$\Phi(X) = \{ \begin{array}{l} 1 \text{ if } \sum_{i=1}^{n} x_i > c \\ \gamma \text{ if } \sum_{i=1}^{n} x_i = c \\ 0 \text{ if } \sum_{i=1}^{n} x_i < c \end{array}$$

such that $E_{\lambda_0}(\Phi(X)) = \alpha$, for some constant c.

Obviously, c is the interger M which satisfies

$$\Pr\left\{\sum_{i=1}^{n} x_i \ge M+1\right\} \le \alpha < \Pr\left\{\sum_{i=1}^{n} x_i \ge M\right\}.$$

Note that this test doesn't depend on λ_1.

(**iii**) we have

$$E_\lambda(\Phi(X)) = \sum_{i=M+1} e^{-n\lambda}(n\lambda)^i/i! + \gamma e^{-n\lambda}(n\lambda)^M/M!,$$

where $\lambda \le \lambda_0$, and

$$\begin{aligned}\frac{d}{d\lambda}E_\lambda\left(\Phi(X)\right) &= -n\sum_{i=M+1} e^{-n\lambda}\left(n\lambda\right)^i/i! + n\sum_{i=M} e^{-n\lambda}\left(n\lambda\right)^i/i! \\ &\quad -n\gamma e^{-n\lambda}\left(n\lambda\right)^M/M! + n\gamma e^{-n\lambda}\left(n\lambda\right)^{M-1}/(M-1)! \\ &= ne^{-n\lambda}\left(n\lambda\right)^M/M! - n\gamma e^{-n\lambda}\left(n\lambda\right)^M/M! + n\gamma e^{-n\lambda}\left(n\lambda\right)^{M-1}/(M-1) \\ &> 0.\end{aligned}$$

This implies $E_\lambda\left(\Phi(X)\right) \leq \alpha$ for $\lambda \leq \lambda_0$.

Thus the test given above is still a UMP $\alpha-$test for the hypotheses $H : \lambda \leq \lambda_0$ v.s. $K : \lambda > \lambda_0$.

Exercise H8.3. Let $X_1, \ldots, X_n$ be independent Poisson Po(λ_1) and $Y_1, \ldots, Y_n$ be independent Po(λ_2), also independent of $X_1, \ldots, X_n$. Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ be the parameter vector, $\boldsymbol{\lambda}_0$ a particular value for this vector (with positive components) and consider hypotheses
$H : \boldsymbol{\lambda} = \boldsymbol{\lambda}_0$
$K : .\boldsymbol{\lambda} \neq \boldsymbol{\lambda}_0$.
Find an asymptotic α-test. **Hint:** find a statistic similar to the χ^2-statistic in the multinomial case (Definition 8.1, p. 86 handout) and its asymptotic distribution under H.

Solution: we have

$$n^{1/2}\left(\bar{X_n} - \lambda_{01}\right) \overset{\mathcal{L}}{\Longrightarrow} N\left(0, \lambda_{01}\right),$$

by the CLT under H, where $\boldsymbol{\lambda}_0 = \left(\lambda_{01}, \lambda_{02}\right)$.

This implies

$$n\left(\bar{X_n} - \lambda_{01}\right)^2/\lambda_{01} \overset{\mathcal{L}}{\Longrightarrow} \chi^2_1,$$

similarly,

$$n\left(\bar{Y_n} - \lambda_{02}\right)^2/\lambda_{02} \overset{\mathcal{L}}{\Longrightarrow} \chi^2_1,$$

under H, then

$$n\left(\bar{X_n} - \lambda_{01}\right)^2/\lambda_{01} + n\left(\bar{Y_n} - \lambda_{02}\right)^2/\lambda_{02} \overset{\mathcal{L}}{\Longrightarrow} \chi^2_2.$$

Thus

$$\begin{aligned}\Phi\left(X,Y\right) &= 1 \text{ if } n\left(\bar{X_n}-\lambda_{01}\right)^2/\lambda_{01}+n\left(\bar{Y_n}-\lambda_{02}\right)^2/\lambda_{02}>\chi^2_{2;1-\alpha} \\ &= 0 \text{ o.w.}\end{aligned}$$

is an asymptotic α-test.

Remark: I hope you guys can derive the asymptotic χ^2 above from likelihood ratio. you can benefit a lot from this.