# Measure Theory and Functional Analysis

## Exercise 1

1. Let $\mathfrak{K}$ be a ring of sets. Let $A, B \in \mathfrak{K}$. Prove that $A \triangle B \in \mathfrak{K}$ and $A \cap B \in \mathfrak{K}$.

2. Prove that a system of sets, closed under operations symmetric difference and intersection, is a ring.

3. Show that a system of sets closed under taking union and itersection is not always a ring.

4. Prove that the set of all intervals in $\mathbb{R}^n$ (i.e. all sets of the form $\{(\xi_1, \dots, \xi_n) \in \mathbb{R}^n : a_i \leqslant \xi_i < b_i\}$ forms a semi-ring.

5. Describe all algebras which can be constructed from the elements of $2^X$, where $X = \{a, b, c\}$.

6. Let $\mathfrak{A}$ be an algebra and $\mu$ is a measure defined on it. Let $A, B \in \mathfrak{A}$. Prove that $|\mu(A) - \mu(B)| \leqslant \mu(A \triangle B)$. (*Hint:* $A \subset A \cup (A \triangle B) = A \cup B$.) Check whether the inequality holds if $\mu$ is an outer measure.

7. Let $X$ be a set, $\mathfrak{A}$ an algebra of its subsets. Let $\tilde{\mathfrak{A}}$ be the $\sigma$-algebra of Caratheodory measurable subsets of $X$. Suppose that $A \subset X$ is such that for any $\epsilon > 0$ there exists $A_\epsilon \in \mathfrak{A}$ such that $\mu^*(A \triangle A_\epsilon) < \epsilon$. Prove that $A \in \tilde{\mathfrak{A}}$.