Measure Theory and Functional Analysis

Exercise 2

1. Let $(A_n)_{n\geq 1}$ be a sequence of measurable sets such that
$$\sum_n \mu(A_n) < \infty.$$
Show that the set of points which belong to infinite number of A_n's has measure zero.
Hint: Prove that the set considered is
$$\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k.$$

2. Prove that if $f(x) \leq g(x)$ for all $x \in A$ then
$$\int_A f d\mu \leq \int_A g d\mu.$$

3. Prove that if $a \leq f(x) \leq b \ (x \in A)), \ \mu(A) < \infty$ then
$$a\mu(A) \leq \int_A f d\mu \leq b\mu(A).$$

4. Prove that if $\mu(A) = 0$ and f is measurable then
$$\int_A f d\mu = 0.$$

5. Let f be an integrable function. Show that
$$(i) \quad \mu(\{|f| \geq a\}) \leq \frac{1}{a} \int |f| d\mu.$$
$$(ii) \quad \mu(\{|f| \geq a\}) = o(\frac{1}{a}) \text{ as } a \to \infty.$$