Measure Theory and Functional Analysis

Exercise 3

I. Compute the integrals

1. $\int\limits_{\mathbb{R}_+} e^{-[x]}dx$, where $[x]$ is the integer part of x;

2. $\int\limits_{[0,1]} \chi_{\mathbb{Q}}(x)dx$, where $\chi_{\mathbb{Q}}$ is the characteristic function of the set of rational numbers;

3. $\int\limits_{[0,1]} f(x)dx$, where $f(x) = \cos x$ if $x \in \mathbb{Q}$ and $f(x) = \sin x$ otherwise.

II. Let $(X, \mathfrak{A}, \mu)$ be a measure space such that $\mu(X) = 1$. Let L^p denote $L^p(X, \mathfrak{A}, \mu)$ for $1 \leq p \leq \infty$. Prove that

1. $L^p \subset L^q$ if $p \geq q \geq 1$;

2. $L^p \cap L^q \subset L^r$ if $1 \leq p \leq r \leq q \leq \infty$ (*Hint:* use the Hölder inequality);

3. If $f \in L^\infty$ then $\|f\|_\infty = \lim\limits_{p\to\infty} \|f\|_p$.