Measure Theory and Functional Analysis

Exercise 4

1. Let Y and Z be subspaces of a vector space X. Show that $Y \cap Z$ is also a subspace of X. Is $Y \cup Z$ always a subspace of X? Give an example.

2. Solve No 11 from the Kreyszig book, Section 2.2.

3. Let X be a normed space. Let $x_n, x, y_n, y \in X$ $(n \in \mathbb{N})$. Prove that

   (a) if $x_n \to x$ then (x_n) is bounded;

   (b) if $x_n \to x, \lambda_n \to \lambda,\ \lambda_n \in \mathbb{C}$ then $\lambda_n x_n \to \lambda x$;

   (c) if $x_n \to x$ then $\|x_n\| \to \|x\|$;

   (d) if $x_n \to x$ and $\|x_n - y_n\| \to 0$ then $y_n \to x$;

   (e) if $x_n \to x$ then $\|x_n - y\| \to \|x - y\|$;

   (f) if $x_n \to x$ and $y_n \to y$ then $\|x_n - y_n\| \to \|x - y\|$.

4. Solve No 8,9 from the Kreyszig book, Section 2.3.

5. Solve No 15 from the Kreyszig book, Section 2.3.

6. Every finite dimensional normed space is a Banach space. Prove it.

7. Let X be a normed space. Suppose that any sequence of closed embedded balls, with the sequence of radii tending to zero, has a non-empty intersection. Prove that X is a Banach space.