Measure Theory and Functional Analysis

Exercise 5

1. Let T be a bounded operator from a normed space X onto a normed space Y. If there exists a positive constant b such that $\|Tx\|_Y \geq \|x\|_X$ for all $x \in X$, show that T is invertible (T^{-1} exists) and that T^{-1} is bounded.

2. Let $T : C[0,1] \to C[0,1]$ be defined by $(Tx)(t) = \int_0^t x(s)ds$ with $D(T) = \{x \in C[0,1];\ x(0) = 0\}$. Find $RanT$ and $T^{-1} : RanT \to C[0,1]$. Is the latter bounded?

3. Prove that the following functional f is bounded and compute its norm.
$$f(x) = \int\limits_{-1}^{1} tx(t)dt, \quad x \in C[-1;1].$$

4. Prove that the following operator A is bounded and compute its norm.
$$(Ax)(t) = \int\limits_{0}^{t} x(\tau)d\tau, \quad x \in C[0;1].$$

5. Let (X, μ) be a σ-finite measure space, $a(x)$ be a bounded measurable function on X. Prove that the following operator A is bounded on every $L^p(X, \mu)$, $1 \leq p \leq \infty$ and compute its norm.
$$(Af)(x) = a(x)f(x).$$

6. Solve No 9 from the Kreyszig book, Section 2.8.

7. Solve No 10 from the Kreyszig book, Section 2.10.

8. Solve No 10,11 from the Kreyszig book, Section 2.10.