Measure Theory and Functional Analysis

Exercise 7

1. Consider the functional $l(x) = \alpha x_1$, where $\alpha \in \mathbb{R}$ is fixed, defined on the subspace $G = \{(x_1, 0) : x_1 \in \mathbb{R}\}$ of the space $\mathbb{R}^2$. Describe all extensions L of the functional l and the extensions with $\|L\| = \|l\|$ for the following norms in the space $\mathbb{R}^2$:

$$\|x\|_\infty = \max\{|x_1|, |x_2|\};$$
$$\|x\|_1 = |x_1| + |x_2|;$$
$$\|x\|_2 = \sqrt{x_1^2 + x_2^2}.$$

2. Let $G = \{x \in C([0;1]) : x(0) = 0\}$. Construct a bounded linear functional on $C([0;1])$ vanishing on G and taking the value 2 on the function $x(t) = t + 1$.

3. Let $E = (C([0;1]), \|\cdot\|_2)$, where

$$\|f\|_2^2 = \int_0^1 |f(x)|^2 dx.$$

   E is known to be incomplete. Show that the assertion of the uniform boundedness theorem is not true in E.

   (Hint: Consider the following sequence sequence of functionals.

$$l_n(f) = \int_0^1 f(x) n \chi_{[1-\frac{1}{n};1]}(x) dx,\ f \in C([0;1]).\ )$$

4. Let X, Y be Banach spaces and $A : X \to Y$ a linear closed operator with $RanA = Y$. Suppose that A is invertible. Show that A^{-1} is continuous.