Measure Theory and Functional Analysis

Exercise 8

1. Find the adjoint to the operator $A : L^2([0;1]) \mapsto L^2([0;1])$ if
   (a) $(Ax)(t) = \int_0^t x(\tau)d\tau$;
   (b) $(Ax)(t) = \int_0^1 (t-\tau)x(\tau)d\tau$.
2. Find the adjoint to the operator $A : l^2 \mapsto l^2$ if
   (a) $Ax = (x_2, x_3, \ldots)$;
   (b) $Ax = (0, x_1, x_2, \ldots)$.
3. Let H be a Hilber space. Let $(A_n), (B_n), A, B$ be bounded operators on H. Suppose that $A_n \to A$ and $B_n \to B$ strongly in H. Prove that
   (a) $A_n B_n \to AB$ strongly in H;
   (b) $A_n^* \to A^*$ weakly in H.
4. Let H be a Hilbert space. Prove that any operator $A \in B(H)$ can be represented in the form
$$A = B + iC$$
   where B and C are self-adjoint.
5. Let X be a Banach space, $A, B \in B(X)$. Let $\lambda \in \rho(A) \cap \rho(B)$. Prove the equality
$$R_\lambda(A) - R_\lambda(B) = R_\lambda(A)(B-A)R_\lambda(B).$$