[523]
UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc., B.A. and M.Sci. (Level III)

MATHEMATICS (MATH 34000)

MEASURE THEORY AND FUNCTIONAL ANALYSIS

Monday 1 JUNE, 1998, 2 15 - 5 15 pm

This paper contains **six** *questions.*
Candidates will be assessed on their **FIVE** *best answers.*

1. (a) State the definition of a measure.

   (b) i. State and prove the continuity property of a measure.

   ii. Let A be a measurable subset of $\mathbb{R}$ such that $\text{meas}(A) < \infty$, where $\text{meas}(A)$ is the Lebesgue measure of A. Show that the function
   $x \mapsto \text{ meas } (A \cap (-\infty, x])$
   is continuous.

   (c) Let μ be a measure. Prove the inequality
   $|\mu(A) - \mu(B)| \leq \mu(A \triangle B)$
   for measurable A and B.

   (d) Let μ be a measure on an algebra $\mathfrak{A} \subset 2^X$. Assume that $\mu(X) = 1$.
   Prove that if for $A_1, ..., A_n \in \mathfrak{A}$
   $$\sum_{i=1}^{n} \mu(A_i) > n - 1 \quad \text{then} \quad \mu(\textstyle\bigcap_{K=1}^{n} A_K) > 0$$
   (Hint: use the fact that $\mu(A_i^c) = 1 - \mu(A_i)$.)

2. (a) State the definition of an L^p-space, $p \geq 1$ (including L^∞).

   (b) i. State and prove Hölder's inequality.

   ii. Show that for $1 \leq p < \infty$ the function
   $p \mapsto p \log \|f\|_p$
   is convex.
   (Assume that $f \in \bigcap_{p \geq 1} L^p$.)

   iii. Let $1 \leq p \leq q \leq r \leq \infty$. Prove that

   $L^p \cap L^q \subset L^r$.

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(c) Let B_1 be a unit ball in $\mathbb{R}^d$ centered at the origin. Let $f \in L^p(B_1)$, for any $p \geq 1$, but $f \notin L^\infty$. Suppose that the following inequality holds

$$\|f\|_{p+1} \leq p^{\frac{1}{p+1}} \|f\|_p^{\frac{p}{p+1}}, \ \forall p \geq 1.$$

Prove that the integral

$$\int_{B_1} e^{\alpha|f(x)|} dx$$

is finite if $\alpha \in (0,1)$.

(Hint: expand $e^{\alpha|f(x)|}$ as a series with respect to powers of $|f(x)|$.)

3. (a) i. Define a linear bounded operator on a normed space.

ii. What is the norm of a linear bounded operator?

(b) Let X and Y be normed spaces and let $T: \mathcal{D}(T) \mapsto Y$ be a linear operator, where $\mathcal{D}(T) \subset X$. Prove that

i. T is continuous if and only if T is bounded.

ii. If T is continuous at a single point then it is continuous.

(c) Prove that the operator $A: C[0,1] \mapsto C[0,1]$ is a linear bounded operator and compute its norm

$$(Ax)(t) = \int_0^t x(\tau)d\tau.$$

($C[0,1]$ is equipped with the standard supremum norm)

(d) Let X be a Banach space, A be a bounded linear operator on X. Define the operator e^A and prove the inequality

$$\|e^A\| \leq e^{\|A\|}.$$

4. (a) State the definition of a Hilbert space.

(b) State and prove Riesz's theorem on the representation of a continuous linear functional on a Hilbert space.

(c) i. State the definition of the adjoint operator in a Hilbert space.

ii. Prove that every bounded linear opeator A can be uniquely represented as the sum $B + iC$, where B and C are self-adjoint operators.

iii. Compute the adjoint to the operator $A: \ L^2[0,1] \mapsto L^2[0,1]$ if $(Ax)(t) = \int_0^1 (t-s)x(s)ds.$

5. (a) Let X be a Banach space, A be a linear bounded operator with $\|A\| < 1$. Prove that $(I - A)^{-1}$ exists and is a bounded linear operator on X. Moreover,

$$\|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|} \ .$$

(b) i. Let X be a Banach space, A, B bounded linear operators from X to X. Let A be invertible and satisfy the following inequality

$$\|B - A\| \leq \|A^{-1}\|^{-1} \ .$$

Prove that B is invertible too, and the inequality

$$\|B^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|B-A\| \ \|A^{-1}\|} \ .$$

holds.

ii. Prove that the set of all continuously invertible operators is an open subset of $B(X)$, the space of all bounded linear operators on a Banach space X.

(c) Let X be a Banach space, A be a compact operator on it, such that $\|I - A\| < 1$. Prove that X is finite dimensional.

(d) Prove that if $X \neq \{0\}$ is a Banach space, $T \in B(X)$ then the spectrum of T is not empty.

6. (a) Define a bounded self-adjoint operator in a complex Hilbert space.

   (b) Prove that all eigenvalues of a self-adjoint operator are real, and that the eigenvectors corresponding to distinct eigenvalues are orthogonal.

   (c) i. Let A be a self-adjoint operator in a complex Hilbert space H, $\lambda = \nu + i\mu$ with ν and μ real.
   Prove the inequality
   $\|(A-\lambda)x\| \geq |\mu| \, \|x\| \qquad \forall x \in H$.

   ii. Conclude from i. that if $\mu \neq 0$ then Ran $(A-\lambda)$ is a closed subspace of H.

   iii. Prove that in ii., in fact,
   Ran $(A-\lambda) = H$.

   iv. Prove that if $\mu \neq 0$ then $\lambda \in \rho(A)$ (the resolvent set of A) and the following estimate holds

$$\|(A-\lambda)^{-1}\| \leq \frac{1}{|\mathrm{Im}\ \lambda|} .$$