UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc., B.A. and M.Sci. (Level III)

**MATHEMATICS** (MATH 34000)

**MEASURE THEORY AND FUNCTIONAL ANALYSIS**

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Thursday 17 APRIL, 1997, 9 30 am - 12 30 pm

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*This paper contains* **seven** *questions.*
*Candidates will be assessed on their* **THREE** *best answers from Section A and* **TWO** *best answers from Section B.*

*Part A*

1. (a) i. Give the definition of an algebra and a $\sigma$-algebra of sets.
   ii. Describe all algebras which can be constructed from the elements of the power set $P(X)$ where $X = \{a, b, c\}$.
   iii. What is a ring of sets?
   Show that a system of sets closed under taking union and intersection is not always a ring.

   (b) i. Define a measure and an outer measure.
   ii. Outline briefly the Carathéodory construction of the extension of a measure.

   (c) i. What is a measurable function?
   ii. Prove that $f$ is measurable if and only if $\arctan f$ is measurable.
   iii. Let $f(x)$ be differentiable everywhere on $[0, 1]$. Prove that $f'(x)$ is Lebesgue measurable on $[0, 1]$.

2. (a) i. State the definition of a normed space and of a Banach space.
   ii. Show that in a Banach space an absolutely convergent series is convergent.

   (b) i. What is the norm of a bounded linear operator acting from a normed space $X$ into a normed space $Y$?
   ii. Prove that the space $B(X, Y)$ of bounded linear operators from a normed space $X$ into a Banach space $Y$ is a Banach space itself.
   iii. Conclude from (ii) that the dual to a normed space is always a Banach space.

   (c) i. What is weak convergence?
   ii. Show that every weak convergent sequence in a normed space is bounded.
   iii. Give an example of a sequence which is weakly convergent but not strongly.

*Cont/...*

3. (a) i. What is an inner product space? What is a Hilbert space? Give examples.
   ii. State and prove the "parallelogram equality".
   iii. Is it possible to obtain the norm in $l^1$ from an inner product?

   (b) i. State and prove Bessel's inequality.
   ii. Let $(e_k)$ be an orthonormal sequence in a Hilbert space $H$. Let $T$ be the linear operator on $H$ defined by

$$Tx = \sum_{k=1}^{\infty} 2^{-k} \quad < x, e_{k+1} > e_k$$

   Prove that $T$ is bounded and compute its norm.

4. (a) State the Hahn-Banach theorem separately for the real and general cases.

   (b) Let $Y$ be a proper closed subspace of a normed space $X$. Let $x_0 \in X \setminus Y$ be arbitrary. Denote the distance $x_0$ to $Y$ by $\delta$, i.e.

$$\delta = \inf_{\tilde{y} \in Y} \|\tilde{y} - X_0\|$$

   i. Show that $\delta > 0$.
   ii. Prove that there exists a linear functional $f \in X'$ such that

   $(a)$ f(x)=0 for any $x \in Y$, $(b)$ f(x$_0$) $= 1$, $(c)$ $\|f\| = \frac{1}{\delta}$.

   (Hint: Consider span $\{x_0, Y\}$ and define the functional by $f(z) = t$ for $z = tx_0 + y, y \in Y$. Use the Hahn-Banach theorem to extend it to the whole of $X$.)

   (c) In $C[-1, 1]$ let us consider the following functional

$$f(x) = \int_{-1}^{1} tx(t)dt.$$

   i. Prove that $f$ is a bounded linear functional and obtain the following estimate

$$\|f\| \leq 1.$$

   ii. Show that in fact the norm of $f$ is 1.
   (Hint: use the following sequence

$$x_n(t) = \begin{cases} 1, & t \in [\frac{1}{n}, 1] \\ nt, & t \in [-\frac{1}{n}, \frac{1}{n}] \\ -1, & t \in [-1, -\frac{1}{n}] \end{cases}$$

   to approximate the discontinuous function

$$sgnt = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$

   on which the value of the functional is easy to compute).

*Part B*

5. (a) Let $X$ be a set, $\mathcal{A}$ an algebra of its subsets. Let $\widetilde{\mathcal{A}}$ be the $\sigma$-algebra of Carathéodory measurable subsets of $X$. Suppose that $A \subset X$ is such that for any $\epsilon > 0$ there exists $A_\epsilon \in \mathcal{A}$ such that $\mu^*(A \triangle A_\epsilon) < \epsilon$.
Prove that $A \in \widetilde{\mathcal{A}}$.

   (b) Let $(f_n)$ be a sequence of functions from $L^p(X, \mu), p \geq 1$ such that

      i. $f_n \to f$ pointwise almost everywhere as $n \to \infty$
      ii. $\|f_n\|_{L^p} \to \|f\|_{L^p}$ as $n \to \infty$.

   Prove that
$$f_n \to f \text{ in } L^p, \text{ as } n \to \infty.$$
   (Hint: introduce the sequence
$$\varphi_n = 2^{p-1}(|f|^p + |f_n|^p) - |f - f_n|^p,$$
   prove that $\varphi_n \geq 0$ for all $n$ and then use Fatou's lemma for this sequence.)

6. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Denote by $B(\mathcal{H})$ the space of bounded linear operators on $\mathcal{H}$, by $\sigma_\infty(\mathcal{H})$ the set of compact operators on $\mathcal{H}$.

   (a) Prove that $\sigma_\infty(\mathcal{H})$ is a closed subspace in $B(\mathcal{H})$.
   (b) Compute the distance between the identity operator ($I_d\, x = x$) and $\sigma_\infty(\mathcal{H})$.

   (Hint: use the fact that if $A \in B(\mathcal{H})$ and $\|A\| < 1$ then the operator $I_d - A$ has the bounded inverse $(I_d - A)^{-1}$.)

7. Let $(e_n)$ be an orthonormal basis in a Hilbert space $\mathcal{H}$.
An operator $A \in B(\mathcal{H})$ is called a Hilbert-Schmidt operator if the quantity
$$\|A\|_2^2 = \sum_{n=1}^{\infty} \|Ae_n\|^2$$
is finite. The set of all Hilbert-Schmidt operators in $\mathcal{H}$ is denoted by $\sigma_2(\mathcal{H})$.
Prove that

   (a) $\|A\|_2$ does not depend on a particular choice of a basis in $\mathcal{H}$,
   (b) $\|A\|_2 = \|A^*\|_2$ ($A^*$ is adjoint to $A$),
   (c) $\|A\| \leq \|A\|_2$,
   (d) i. $\|A\|_2$ is a norm on $\sigma_2(\mathcal{H})$,
       ii. $\sigma_2(\mathcal{H})$ is a linear subspace of $B(\mathcal{H})$,
   (e) $(A, B) = \sum_{n=1}^{\infty}(Ae_n, Be_n)$ defines an inner product on $\sigma_2(\mathcal{H})$,

(f) $\sigma_2(\mathcal{H})$ is a Hilbert space,

(g) If $A \in B(\mathcal{H})$ and $B \in \sigma_2(\mathcal{H})$ then $AB \in \sigma_2(H)$ and $BA \in \sigma_2(\mathcal{H})$ and

$$\|AB\|_2 \leq \|A\|_2 \; \|B\|,$$

$$\|BA\|_2 \leq \|A\|_2 \; \|B\|.$$