Measure Theory and Functional Analysis

Solutions to Exercise 1

1. If $A, B \in \mathfrak{K}$ then $A \setminus B, B \setminus A \in \mathfrak{K}$. Therefore $A \triangle B = (A \setminus B) \cup (B \setminus A) \in \mathfrak{K}$. Also, $A \cap B = A \setminus (A \setminus B) \in \mathfrak{K}$.

2. One has to prove that
$$(A, B \in \mathfrak{K}) \Longrightarrow (A \cup B \in \mathfrak{K} \text{ and } A \setminus B \in \mathfrak{K}).$$
Given $A, B \in \mathfrak{K}$ it follows that $A \triangle B, A \cup B \in \mathfrak{K}$. But
$$A \cup B = (A \triangle B) \triangle (A \cap B) \text{ and } A \setminus B = A \triangle B(A \cap B).$$

3. Consider a system of sets containing two sets only : A and B such that $B \subset A$. Then this system contains the union $A \cup B = A$ and the intersection $A \cap B = B$, but it does not contain the difference $A \setminus B$.

4. One has to prove that if A, B are intervals in $\mathbb{R}^n$ then $A \cap B$ is an interval in $\mathbb{R}^n$ and $A \setminus B$ is a disjoint union of intervals in $\mathbb{R}^n$.

   If $A = \displaystyle\prod_{i=1}^{n}[a_i, b_i)$ and $B = \displaystyle\prod_{i=1}^{n}[c_i, d_i)$ then $A \cap B = \displaystyle\prod_{i=1}^{n}([a_i, b_i) \cap [c_i.d_i))$ an is an interval in $\mathbb{R}^n$.

   In order to prove that $A \setminus B$ is equal to a finite union of disjoint intervals in $\mathbb{R}^n$ let us denote by $\alpha_{i1}, \alpha_{i2}, \alpha_{i3}, \alpha_{i4}$ the rearrangement of a_i, b_i, c_i, d_i in the non-decreasing order. Denote by C_j, $1 \leq j \leq 3^n$, the intervals which are obtained as intersection of the strips $\{\alpha_{im} \leq \xi_i < \alpha_{i(m+1)}\}$. Then
$$A = \bigsqcup_{j=1}^{3^n} (A \cap C_j)$$
   is the representation of A as a finite union of disjoint intervals in $\mathbb{R}^n$, one of which is $A \cap B$, which leads to the required representation for $A \setminus B$.

5. $\mathfrak{A}_1 = \{\varnothing, X\}$;
   $\mathfrak{A}_2 = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$;
   $\mathfrak{A}_3 = \{\varnothing, X, \{a\}, \{b,c\}\}$;
   $\mathfrak{A}_4 = \{\varnothing, X, \{b\}, \{a,c\}\}$;
   $\mathfrak{A}_5 = \{\varnothing, X, \{c\}, \{a,b\}\}$.

6. Since $A \subset (A \triangle B) \cup B$, by the monotonocity and semi-additivity of a measure
$$\mu(A) \leq \mu(A \triangle B) + \mu(B).$$
   Analogously
$$\mu(B) \leq \mu(A \triangle B) + \mu(A).$$
   The properties of monotonicity and semi-additivity hold also for the outer measure μ^*, therefore the inequality holds as well.

7. Since any outer measure is semi-additive, it suffices to prove that for any $E \subset X$ one has
$$\mu^*(E) \geqslant \mu^*(E \cap A) + \mu^*(E \cap A^c). \tag{1}$$
   Since $\mathfrak{A} \subset \tilde{\mathfrak{A}}$, one has
$$\mu^*(E \cap A_\varepsilon) + \mu^*(E \cap A_\varepsilon^c) \leqslant \mu^*(E). \tag{2}$$
   It follows from the result of the previous exercise (for the outer measure) and from the monotonicity of μ^* that
$$|\mu^*(E \cap A_\varepsilon) - \mu^*(E \cap A)| \leqslant \mu^*((E \cap A_\varepsilon) \triangle (E \cap A)) \leqslant \mu(A_\varepsilon \cap A) < \varepsilon.$$
   Therefore, $\mu^*(E \cap A_\varepsilon) > \mu^*(E \cap A) - \varepsilon$. Analogously, $\mu^*(E \cap A_\varepsilon^c) > \mu^*(E \cap A^c) - \varepsilon$. Using (2), one obtains
$$\mu^*(E) > \mu^*(E \cap A) + \mu^*(E \cap A^c) - 2\varepsilon.$$
   Since $\varepsilon > 0$ is arbitrary, (1) follows.