Measure Theory and Functional Analysis

Solution to Exercise 2

1. It is clear that if $x \in \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k$ then x belongs to infinitely many A_n's. Now let x belongs to infinitely many A_n's. Then for any n there is a $k \geq n$ such that $x \in A_k$. So we have proved that the set considered is
$$A = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k.$$
For every $n \in \mathbb{N}$ one has
$$\mu\left(\bigcup_{k \geq n} A_k\right) \leq \sum_{k \geq n} \mu(A_k)$$
and consequently
$$\mu(A) = \lim_n \mu\left(\bigcup_{k \geq n} A_k\right) \leq \lim_n \sum_{k \geq n} \mu(A_k) = 0.$$

2. By the definition of the inegral we have
$$\int_A f d\mu = \sup I(\varphi)$$
where the sup is taken over the set of simple functions such that $\varphi \leq f$ on A. Analogously for g. Since $\{\varphi : \ \varphi \leq f\} \subset \{\varphi : \ \varphi \leq f\}$ the result follows from the property of sup.

3. For every simple φ such that $\varphi \leq f$ on A we have $\varphi \leq b\chi_A$. Therefore $\int_A \varphi d\mu \leq b\mu(A)$. Taking sup over all such φ's we obtain $\int_A f d\mu \leq b\mu(A)$. The oposite inequality is obtained in the same way using that $-f(x) \leq -a$.

4. For φ as above $\int_A \varphi d\mu = 0$. Taking sup over all such φ's we obtain the result.

5. (i)

$$\mu(\{|f| \geq a\}) = \int_{\{|f|\geq a\}} d\mu \leq \int_{\{|f|\geq a\}} \frac{|f|}{a} d\mu \leq \frac{1}{a} \int |f| d\mu$$

   (ii) From (i) if $a \to \infty$ then $\mu(\{|f| \geq a\}) \to 0$ which proves the result.