Measure Theory and Functional Analysis

Solutions to Exercise 3

I.

1.
$$\int_{\mathbb{R}_+} e^{-[x]}dx = \sum_{k=0}^{\infty} \int_k^{k+1} e^{-[x]}dx = \sum_{k=0}^{\infty} e^{-k} = \frac{e}{e-1};$$

2. $\chi_{\mathbb{Q}}(x) \equiv 0$ a.e. so $\int_0^1 \chi_{\mathbb{Q}}(x)dx = 0$;

3. $f(x) = \sin x$ a.e. so $\int_0^1 f(x)dx = 1 - \cos 1$.

II.

1. It is enough to show that $\|f\|_p \geq \|f\|_q$ for every simple function f. By the Hölder inequality,
$$\int_X |f|^q d\mu \leq (\int_X |f|^p d\mu)^{q/p} (\int_X d\mu)^{(p-q)/p} = \|f\|_p^q.$$

2. It is enough to show that $\|f\|_r \geq \|f\|_q^{\phi} \|f\|_p^{1-\phi}$, where $\phi \in [0,1]$ is such that $r^{-1} = \phi p^{-1} + (1-\phi)q^{-1}$. Let $\alpha \in [0,1]$ be such that $r = \alpha p + (1-\alpha)q$. Then
$$\int_X |f|^r d\mu = (\int_X |f|^p d\mu)^{\alpha} (\int_X |f|^q d\mu)^{1-\alpha},$$
by the Hölder inequality. So
$$\|f\|_r \leq \|f\|_p^{\alpha \frac{p}{r}} \|f\|_q^{(1-\alpha)\frac{q}{r}}.$$
Finally observe that $\phi = \alpha \frac{p}{r}$.

3. It is enough to prove the statement for the case $\|f\|_\infty = 1$. First, by 1, $p \mapsto \|f\|_p$ is non-decreasing and bounded above with 1. So $c(f) = \lim_{p\to\infty} \|f\|_p$ does exist and $c(f) \leq 1$. Since $\|f\|_\infty = 1$, for every $\epsilon > 0$, there exists a measurable $E_\epsilon \subset X$, $\mu(E_\epsilon) > 0$, such that $|f|(x) \geq 1-\epsilon$ at almost all $x \in E_\epsilon$. Then $\|f\|_p \geq (1-\epsilon)\mu(E_\epsilon)^{1/p} \to 1-\epsilon$ as $p \to \infty$. Hence $1-\epsilon \leq c(f) \leq 1$ for every $\epsilon > 0$. So $c(f) = 1$.