Measure Theory and Functional Analysis

Solutions to Exercise 4

1. (a) Let Y, Z be subspaces of a vector space X. Set some arbitrary $x_1, x_2 \in Y \cap Z$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Denote $x = \lambda_1 x_1 + \lambda_2 x_2$. Then $x \in Y$ and $x \in Z$, so $x \in Y \cap Z$.

   (b) In general, the union of two subspaces is not a subspace. For example, let $X = \mathbb{R}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$, $Y = \{(x_1, 0) : x_1 \in \mathbb{R}\}$ and $Z = \{(0, x_2) : x_2 \in \mathbb{R}\}$. Obviously, Y and Z are subspaces of X. Let now $y = (1, 0)$ and $z = (0, 1)$. Then $y \in Y$, $z \in Z$ and so both vectors belong to $Y \cup Z$. However, $y + z = (1, 1)$ does not belong neither Y nor Z. Hence, $Y \cup Z$ is not a subspace of X.

2. We have to show that, for all x, y and α such that $\|x\| \le 1$, $\|y\| \le 1$ and $0 \le \alpha \le 1$, $\|\alpha x + (1-\alpha)y\| \le 1$. By the triangle inequality,

$$\|\alpha x + (1-\alpha)y\| \le \alpha\|x\| + (1-\alpha)\|y\| \le \alpha + (1-\alpha) = 1.$$

3. (a) If $x_n \to x$ then there exists N such that $\|x_n - x\| < 1$ for all x_n, $n > N$. Hence $\|x_n\| < 1 + \|x\|$ for all x_n, $n > N$, by the triangle inequality. On the other hand, $\max\{\|x_n\| : n \le N\} =: M$ is finite, as the maximum over a finite quantity of numbers. Thus, $\|x_n\| \le M + 1 + \|x\|$.

   (b) We have to show that $\|\lambda_n x_n - \lambda x\| \to 0$ as $n \to \infty$. By the triangle inequality,

$$\|\lambda_n x_n - \lambda x\| \le \|\lambda_n x_n - \lambda x_n\| + \|\lambda x_n - \lambda x\|.$$

   Consider the summands in the right hand side above separately.

i. $\|\lambda_n x_n - \lambda x_n\| = |\lambda_n - \lambda| \, \|x_n\|$. Since $x_n \to x$, there exists M such that $\|x_n\| < M$ for all n, as it was shown in (a). Therefore $|\lambda_n - \lambda| \, \|x_n\| \leq M|\lambda_n - \lambda| \to 0$ since $|\lambda_n \to |\lambda$.

ii. $\|\lambda x_n - \lambda x\| = |\lambda| \|x - x_n\| \to 0$ since $x_n \to x$.

(c) $\|x\| \leq \|x_n\| + \|x - x_n\|$ and $\|x_n\| \leq \|x\| + \|x - x_n\|$, by the triangle inequality. Hence $|\|x\| - \|x_n\|| \leq \|x - x_n\|$. Thus, $\|x_n\| \to \|x\|$ since $x_n \to x$.

(d) By the triangle inequality, $\|x - y_n\| \leq \|x - x_n\| + \|x_n - y_n\| \to 0$ since $x_n \to x$ and $\|x_n - y_n\| \to 0$.

(e) $(x_n - y) \to (x - y)$ since $x_n \to x$. Hence $\|x_n - y\| \to \|x - y\|$, by (c).

(f) It is enough to show that $(x_n - y_n) \to (x - y)$, becouse of (c). Now, $\|(x_n - y_n) - (x - y)\| \leq \|x_n - x\| + \|y_n - y\|$, by the triangle inequality. Thus, $(x_n - y_n) \to (x - y)$ since $x_n \to x$ and $y_n \to y$.

4. (a) Kreyszig's, 2.3, No 8.

Let (x_n) be a Cauchy sequence. Let $N_1 \geq 1$ be such that $\|x_n - x_m\| \leq 2^{-1}$ for all $n, m \geq N_1$, $N_2 \geq N_1$ be such that $\|x_n - x_m\| \leq 2^{-2}$ for all $n, m \geq N_2$, ... , $N_k \geq N_{k-1}$ be such that $\|x_n - x_m\| \leq 2^{-n}$ for all $n, m \geq N_k$ and so forth. Define a series (y_n) by $y_1 = x_{N_1}$ and $y_n = x_{N_n} - x_{N_{n-1}}$ for all $n \geq 2$. Then $\|y_1\| \leq 2^{-1}$ and $\|y_n\| \leq 2^{-n+1}$ for all $n \geq 2$. Therefore (y_n) is absolutely convergent and so it has a limit x. Thus $x_{N_n} = \sum_{k=1}^{n} y_k \to x$ as $n \to \infty$. Now, for an $\epsilon > 0$ fix m such that $2^{-m+1} < \epsilon$ and $\|x_{N_m} - x\| < (\epsilon/2)$. Then, for all $n \geq N_m$, $\|x_n - x\| \leq \|x_n - x_{N_m}\| + \|x_{N_m} - x\| < \epsilon$, by the triangle inequality. Thus, $x_n \to x$.

(b) Kreyszig's, 2.3, No 9.

Let (x_n) be an absolutely covergent series. Consider the sequence (s_n) of its partial sums: $s_n := \sum_{k=1}^{n} x_k$. $\|s_n - s_m\| = \|\sum_{k=m}^{n} x_k\| \leq \sum_{k=m}^{n} \|x_k\|$ for $m \leq n$. So (s_n) is a Cauchy sequence since (x_n) converges absolutely. Hence $s_n \to s$ as the space is complete.

5. Kreyszig's, 2.3, No 15.

We check the properties of a norm, one by one. Let $\alpha \in \mathbb{R}$, $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$.

(a) Homogeneity.
$\|\alpha x\| = \max\{\|\alpha x_1\|_1, \|\alpha x_2\|_2\} = |\alpha| \max\{\|x_1\|_1, \|x_2\|_2\} = |\alpha|\ \|x\|$.

(b) The triangle inequality.

$$\|x+y\| = \max\{\|x_1+y_1\|_1, \|x_2+y_2\|_2\} \leq \max\{\|x_1\|_1+\|y_1\|_1, \|x_2\|_2+\|y_2\|_2\}.$$

Now we are left to check that $\max\{a+b, c+d\} \leq \max\{a,c\} + \max\{b,d\}$ for all $a, b, c, d \geq 0$.
If $a \geq c$, $b \geq d$ then $\max\{a+b, c+d\} = a+b$.
If $a < c$, $b \geq d$ then $\max\{a+b, c+d\} < c+b$.
If $a \geq c$, $b < d$ then $\max\{a+b, c+d\} < a+d$.
If $a < c$, $b < d$ then $\max\{a+b, c+d\} = c+d$.

(c) Positivity.
Let $\|x\| = 0$. Then $\max\{\|x_1\|_1, \|x_2\|_2\} = 0$ and so $x_1 = 0, x_2 = 0$.

6. First we show that all norms are equivalent on a finite dimensional space X. Let n be the dimension of X. Fix a basis $\{e_1, e_2, \dots, e_n\}$ in X and $\{f_1, f_2, \dots, f_n\} \subset X^*$ such that $f_i(e_j) = \delta_{ij}$. Every $x \in X$ is of the form $x = \sum_{k=1}^n \xi_k e_k$ where $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ uniquely corresponds to x. Consider on X another norm $\|\cdot\|_*$,

$$\|x\|_* = \max_{1\leq k\leq n} |\xi_k| = \max_{1\leq k\leq n} |f_k(x)|.$$

(The proof of the fact that it is a norm ia the same as in 5.) $\|x_m\| \to 0$ implies that $\|x_m\|_* \to 0$ since $f_k \in X^*$, $k = 1, \dots, n$. On the other hand, if $\|x_m\|_* \to 0$ with $x_m = \sum_{k=1}^n \xi_k^{(m)} e_k$ then $\xi_k^{(m)} \to 0$ for all $1 \leq k \leq n$ and hence $\|x_m\| \to 0$. Therefore $\|\cdot\|_*$ is equivalent to $\|\cdot\|$. Now let (x_m) be a Cauchy sequence, $x_m = \sum_{k=1}^n \xi_k^{(m)} e_k$. Then $(\xi_k^{(m)})$ is a Cauchy sequence and hence $\xi_k^{(m)} \to \xi_k$ for all $1 \leq k \leq n$. Let $x = \sum_{k=1}^n \xi_k e_k$. Obviously, $\|x - x_m\|_* \to 0$. So X is complete.

7. Let (x_n) be a Cauchy sequence. Fix $N_1 \geq 1$ such that $\|x_n - x_m\| < 2^{-1}$ for $n, m \geq N_1$, $N_2 \geq N_1$ such that $\|x_n - x_m\| < 2^{-2}$ for $n, m \geq N_2$, ..., $N_k \geq N_{k-1}$ such that $\|x_n - x_m\| < 2^{-k}$ for $n, m \geq N_k$ and so on. Then the sequence $B_k = \{x : \|x - x_{N_k}\| \leq 2^{-k+1}\}$ is a sequence of closed embedded balls with the sequece of vanishing radii and there

are at most finitely many elements of (x_n) outside of every B_k. By the assumption, $\cap B_k$ is non-empty and, by the construction, every point of the intersection is a limit point of (x_n). However, (x_n) can have one limit point at most, as a Cauchy sequence. So (x_n) is convergent. Thus we have proved that an arbitrary Cauchy sequence is convergent. Hence X is complete.