Measure Theory and Functional Analysis

Solutions of Exercise 5

1. If $Tx_1 = Tx_2$ then $\|x_1 - x_2\|_X \le b^{-1}\|Tx_1 - Tx_2\|_Y = 0$. So T is one-to-one and hence invertible. T^{-1} is bounded since $\|y\|_Y \ge b\|T^{-1}y\|_X$.

2. $RanT$ consists of elements of $C[0,1] \cap C^1(0,1)$ which are differentiable from the right at 0 and from the left at 1 and vanish at 0. The (left) inverse of T is $T^{-1}u = \dot{u}$. To show that it is unbounded, consider sequence $x_n(t) = t^n$. $\|x_n\| = 1$ while $\|T^{-1}x_n\| = n,\ n = 1,\ 2, \ldots$.

3. $$|f(x)| \le \int_1^1 |tx(t)|dt \le \|x\| \int_1^1 |t|dt = \|x\|.$$

   So f is bounded and $\|f\| \le 1$. To show that $\|f\| = 1$, consider sequence x_n, $x_n(t) \equiv -1$ for $t \in [-1, -1/n]$, $x_n(t) = nt$ for $t \in [-1/n, 1/n]$ and $x_n(t) \equiv 1$ for $t \in [1, 1/n]$. Then $\|x_n\| = 1$ while $f(x_n) = 1 - 3^{-1}n^{-2} \to 0$ as $n \to \infty$.

4. $$\|Ax\| = \max_t \left|\int_0^t x(\tau)d\tau\right| \le \int_0^1 \|x\|d\tau = \|x\|.$$

   So A is bounded and $\|A\| \le 1$. Now let $x(t) \equiv 1$. Then $\|x\| = 1$, $(Ax)(t) = t$ and so $\|Ax\| = 1$. Hence $\|A\| = 1$.

5. Let $\|a\|_\infty = m$. Then $|a(x)f(x)| \le m|f(x)|$ for μ-a.a $x \in X$. Therefore A is bounded as an operator on $L^p(X,\mu), 1 \le p \le \infty$ and $\|A\| \le m$. Now we prove that $\|A\| = m$. Since $\|a\|_\infty = m$, for every $n \in \mathbb{N}$ there exists a finite measure set $E_n \subset X$ such that $m - (1/n) < |a(x)| \le m$

for a.a $x \in E_n$. Denote the indicator of E_n by χ_n. Obviously, $\|\chi_n\|_p = (\mu(E_n))^{1/p}$ and

$$(m - (1/n))(\mu(E_n))^{1/p} < \|A\chi_n\|_p \le m(\mu(E_n))^{1/p}$$

for $1 \le p \le \infty$. Thus, $\|A\| = m$.

6. Assume the contrary: there are $x \in X$, $y_1, y_2 \in \mathfrak{N}(f)$ and constants $\alpha_1 \ne \alpha_2$ such that $\alpha_1 x_0 + y_1 = x = \alpha_2 x_0 + y_2$. Then $(\alpha_1 - \alpha_2)f(x_0) = f(y_2 - y_1) = 0$. This contradicts to the assumption that $\alpha_1 \ne \alpha_2$, as $f(x_0) \ne 0$. So $\alpha_1 = \alpha_2$ and hence $y_1 = y_2$.

7. Let $dimZ = k < n$ and set a basis $e_1, e_2, \ldots, e_k$ in Z. Since x_0 belongs to $X - Z$, it is linearly independent of $e_1, e_2, \ldots, e_k$. Consider the linear hall Z_1 of $e_1, e_2, \ldots, e_k, e_{k+1}$, where $e_{k+1} = x_0$. If $n = k + 1$ then $X = Z_1$ and we define f by $f(e_i) = 0$, $i = 1, 2, \ldots, k$ and $f(k+1) = 1$. If $n > k+1$ consider the subspace $Z_0 = X - Z_1$. It is a proper subspace of dimension $n-k-1$ and so we can set a basis $e_{k+2}, e_{k+3}, \ldots, e_n$ in Z_0. The set $e_1, e_2, \ldots, e_n$ is a basis in X. Now we define f by $f(e_{k+1}) = 1$ and $f(e_i) = 0, i \ne k + 1$.

8. Let $e_1, e_2, \ldots, e_k, \ldots$ be a countable subset of a Hamel basis $\{e_\alpha\}$ in X. WLOG we can assume that the basis is normalized. Now let us define the linear functional f on X by $f(e_k) = k$ on the chosen subset of the basis and $f(e_\alpha) = 0$ on the rest of the basis. f is unbounded since, for every $n \in \mathbb{N}$, $f(e_n) = n = n\|e_n\|$. Hence f is not in X' though it is in X^*. Now let $y \in Y, \|y\|_Y = 1$. Then $T : X \to Y$, $Tx = f(x)y$ is unbounded since $\|Tx\|_Y = |f(x)|$.