Measure Theory and Functional Analysis

Solutions to Exercise 6

1. (a) Consider the following linear functional f on H.

$$f(x) = \int_0^1 x(t)dt.$$

By the Schwartz inequality,

$$|f(x)| = |\int_0^1 x(t)dt| \leq |\int_0^1 |x(t)|^2 dt|^{1/2} = \|x\|.$$

So f is a bounded linear functional and M is its null space. Hence M is a closed subspace in H.

   (b) $M^{\perp} = \mathfrak{M}(f)^{\perp}$, so $dim M^{\perp} = 1$. Obviously, $1 \perp M$. Hence $M^{\perp} = \{x \in H : x(t) \equiv const.\}$.

   (c) The projection of x onto $\mathfrak{M}(f)$ is $x - f(x)1$, where $1(t) \equiv 1$. Hence the distance of $x(t) = t^2$ from M is $|f(x)| = 1/3$.

2.

$$0 = \|x\|^2 - \|y\|^2 = \langle x + y, x - y \rangle.$$

Geometrically it means that the diagonals of a are perpendicular. If X is complex then

$$0 = \|x\|^2 - \|y\|^2 = Re\langle x + y, x - y \rangle.$$

3. Assume that $\|x - \alpha y\| = \|x + \alpha y\|$ forall α. Then, by the previous , $4Re\alpha\langle y, x \rangle = 0$. Hense $x \perp y$.

4. (a) No 9.

$\|x_0\| = \sqrt{2}$. So we set $e_0(t) \equiv (1/\sqrt{2})$. $\langle x_0, x_1 \rangle = 0$ and $\|x_1\| = \sqrt{(2/3)}$. Hence we set $e_1(t) = \sqrt{(3/2)}t$. $\langle x_0, x_2 \rangle = 2/3$ and $\langle x_1, x_2 \rangle = 0$. $\|x_2 - (2/3)x_0\| = \sqrt{(2/5)}$. Therefore we set $e_2(t) == \sqrt{(5/2)}(t^2 - (2/3))$.

(b) No 10.

$\|x_1\| = \sqrt{(2/5)}$. So we set $e_1(t) = \sqrt{(5/2)}t^2$. $\langle x_1, x_2 \rangle = 0$. $\|x_2\| = \sqrt{(2/3)}$. So we set $e_2(t) = \sqrt{(3/2)}t$. $\langle x_1, x_3 \rangle = 2/3$. $\langle x_2, x_3 \rangle = 0$. $\|x_3 - (2/3)x_1\| = \sqrt{(58/45)}$. Thus we set $e_3(t) = \sqrt{(45/58)}(1 - (2/3)t^2)$.