Measure Theory and Functional Analysis

Solutions to Exercise 7

1. Let L be a linear continuous functional on $\mathbb{R}^2$. Then there exist $a, b \in \mathbb{R}$ such that for all $x \in \mathbb{R}^2$

$$L(x) = ax_1 + bx_2$$

. It is easy to see now that the functional L extends l iff $a = \alpha$. Let us compute the norm of L in the spaces $(\mathbb{R}^2, ||\cdot||_2)$, $(\mathbb{R}^2, ||\cdot||_1)$, $(\mathbb{R}^2, ||\cdot||_\infty)$. For that reason let us estimate $|L(x)|$ in three different ways.

$$|L(x)| = ||ax_1 + bx_2| \le \sqrt{a^2 + b^2}\sqrt{x_1^2 + x_2^2} = \sqrt{a^2 + b^2}||x||_2$$

It is easy to see that $||L||_2 = \sqrt{a^2 + b^2}$.

$$|L(x)| = ||ax_1 + bx_2| \le max\{|a|, |b|\}(|x_1| + |x_2|) = max\{|a|, |b|\}||x||_1$$

Also, $||L||_1 = max\{|a|, |b|\}$.

$$|L(x)| = ||ax_1 + bx_2| \le (|a| + |b|)max\{|x_1|, |x_2|\} = (|a| + |b|)||x||_\infty$$

One can easily verify that $||L||_\infty = (|a| + |b|)$

Let L be an extension of l.Then

$$L(x) = \alpha x_1 + bx_2$$

Consider all three norms of the functional.

$$||L||_2 = \sqrt{\alpha^2 + b^2},$$

therefore $||L||_2 = ||l||_2 = |\alpha|$ iff $b = 0$.

$$||L||_1 = max\{|\alpha|, |b|\},$$

hence $||L||_1 = ||l||_1 = |\alpha|$ iff $|b| \leq |\alpha|$

$$||L||_\infty = |\alpha| + |b|,$$

it follows that $||L||_\infty = ||l||_\infty$ iff $b = 0$ Thus, in the space $(\mathbb{R}^2, ||\cdot||_1)$ the functional l can be extended with its norm being preserved.

2. Consider the following functional L on $C([0;1])$ as follows.

$$L(x) = 2x(0).$$

It is easy to show that L is a linear continuous functional on $C([0;1])$. Let $x_0(t) = t + 1$.Then

$$L(x_0) = 2x_0(0) = 2.$$

3. Let us consider the sequence of functionals from the hint.$\forall n$, $n \geq 1$, l_n is a linear continuous functional on $L^2([0;1])$ as $\forall n \geq 1$ the function $g(x) = n\chi_{[1-\frac{1}{n},1]}(x)$ belongs to $L^2([0;1])$.

   Denote by $||\cdot||$ the norm in the space $C([0;1])$. Let us estimate $|l_n(f)|$ for each $f \in C([0;1])$.

$$|l_n(f)| \leq \int_{1-\frac{1}{n}}^{1} n|f(x)|dx \leq ||f||$$

   Thus, for each $f \in C([0;1])$ the sequence $\{l_n(f), n \geq 1\}$ is bounded. However, $||l_n|| = \sqrt{n} \to \infty$ as $n \to \infty$.

4. By the closed graph theorem a linear operator is continuous iff it is closed and its domain is the whole space. Consider the operator A^{-1}. $Dom(A^{-1}) = RanA = Y$, i.e. it is defined on the whole space. Moreover, the operator A^{-1} is closed since the operator A is closed. Therefore the operator A^{-1} is continuous.