Measure Theory and Functional Analysis

Solutions to Exercise 8

1. (a) Let $x, y \in L^2([0;1])$. Then

$$a\langle Ax, y\rangle = \int_0^1 \int_0^t x(\tau) d\tau \overline{y(t)} dt = \int_0^1 d\tau x(\tau) \overline{\int_\tau^1 y(t) dt} = \langle x, A^* y\rangle.$$

Therefore

$$(A^* y)(\tau) = \int_\tau^1 y(t) dt.$$

(b) Analogously

$$(A^* y)(t) = \int_0^1 (t - \tau) y(\tau) d\tau.$$

2. (a) Let $x, y \in l^2$. Then

$$\langle Ax, y\rangle = \sum_{i=1}^{\infty} x_{i+1} \overline{y_i} = x_1 \cdot 0 + x_2 \cdot \overline{y_1} + x_3 \cdot \overline{y_2} + \ldots = \langle x, A^* y\rangle.$$

Therefore the adjoint is

$$A^* y = (0, y_1, y_2, \ldots).$$

(b) Analogously,

$$A^* y = (y_2, y_3 \ldots.).$$

3. (a) From the convergence of (A_n) it follows that there is a constant C such that $\|A_n\| \leq C$, Let $x \in H$. We have

$$\|A_n B_n x - ABx\| = \|A_n B_n x - A_n Bx + A_n Bx - ABx\| \leq \|A_n(B_n x - Bx)\|$$

$$+\|A_nBx - ABx\| \leq C\|B_nx - Bx\| + \|A_nBx - ABx\| \to 0.$$

(b) Let $x, y \in H$. Then

$$\langle A_n^* x, y\rangle = \langle x, A_n y\rangle \to \langle x, Ay\rangle = \langle A^* x, y\rangle.$$

4. Set
$$B = \frac{1}{2}(A + A^*),\ C = \frac{1}{2i}(A - A^*).$$
Using properties of the adjoint operator we have
$$B^* = \left(\frac{A + A^*}{2}\right) = \frac{A^* + A^{**}}{2} = \frac{A + A^*}{2} = B,$$
$$C^* = \left(\frac{A - A^*}{2i}\right) = \frac{A^* - A^{**}}{-2i} = \frac{A - A^*}{2i} = C.$$

5. Let us multiply the left hand side of the equality by $A_\lambda I$. Then we obtain
$$(A - \lambda I)[R_\lambda(A) - R_\lambda(B)] = I - (B - \lambda I)R_\lambda(B) + (B - A)R_\lambda(B)$$
$$= B - A)R_\lambda(B).$$
Since $A - \lambda I$ is an isomorphism of X this proves the required equality.