Solutions to the exam paper on Measure Theory and Functional Analysis, 1996/97

Dr. Vitali Liskevich

1 Question 1.

(a)

(i). An algebra of sets is a non-empty subset $\mathfrak{A}$ of 2^X (power set of X) satisfying

1) $(A \in \mathfrak{A}) \Longrightarrow (A^c \in \mathfrak{A})$;

2) $(A, B \in \mathfrak{A}) \Longrightarrow (A \cup B \in \mathfrak{A})$;

3) $X \in \mathfrak{A}$.

An algebra $\mathfrak{A}$ of sets is called a σ-algebra if for any sequence of sets (A_n) from $\mathfrak{A}$ their union belongs to $\mathfrak{A}$, that is $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{A}$.

(ii). $\mathfrak{A}_1 = \{\varnothing, X\}$;

$\mathfrak{A}_2 = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$;

$\mathfrak{A}_3 = \{\varnothing, X, \{a\}, \{b,c\}\}$;

$\mathfrak{A}_4 = \{\varnothing, X, \{b\}, \{a,c\}\}$;

$\mathfrak{A}_5 = \{\varnothing, X, \{c\}, \{a,b\}\}$.

(iii). A ring of sets is a non-empty subset of 2^X closed with respect to the operations $\cup$ and $\setminus$. Consider a system containing two sets $\{A, B\}$ such that $A \subset B$. Then $A \cap B = A \in \{A, B\}$, $A \cup B = B \in \{A, B\}$. But $A \setminus B \notin \{A, B\}$ and $B \setminus A \notin \{A, B\}$.

(b)

(i). Let $(X, \mathfrak{A})$ be a set with an algebra of subsets. A measure μ is a function $\mu : \mathfrak{A} \mapsto \mathbb{R}_+ \cup \{\infty\}$ such that

1) $\mu(A) \geq 0 \ \forall A \in \mathfrak{A}, \ \mu(\varnothing) = 0.$
2) $\mu(\bigsqcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j)$ for $(A_j)_{j=1}^{\infty} \subset \mathfrak{A}$, disjoint.

(ii). Let X be a set with an algebra $\mathfrak{A}$ of its subsets, and μ a measure on $\mathfrak{A}$. For any $A \subset X$ define

$$\mu^*(A) = \inf \sum_{j=1}^{\infty} \mu(E_j),$$

where inf is taken over all countable coverings of A: $\bigcup_{j=1}^{\infty} E_j \supset A$.

μ^* is called *outer measure*, and it is defined on 2^X. However it may be not σ-additive.

By definition a set $A \subset X$ is called measurable if

$$\mu^*(E) = \mu^*(A \cap E) + \mu^*(A^c \cap E)$$

for any $E \subset X$.

One proves that the set of all measurable sets, $\tilde{\mathfrak{A}}$, is a σ-algebra of subsets of X, and μ^* restricted to $\tilde{\mathfrak{A}}$ is a measure, i.e. it is possible to prove the σ-additivity property. Moreover, $\mu^* = \mu$ on $\mathfrak{A}$, i.e. μ^* is an extension of μ. This σ-algebra, of course, contains the minimal σ-algebra generated by $\mathfrak{A}$.

(c)

(i). A function $f : X \mapsto \mathbb{R}$ defined on a measurable space $(X, \mathfrak{A})$ with a σ-algebra $\mathfrak{A}$, is called measurable if for any $a \in \mathbb{R}$ the set

$$\{x \in X : \ f(x) > a\} \in \mathfrak{A}.$$

(ii). If $f(x)$ is measurable then $\arctan f(x)$ is measurable as a composition $\varphi \circ f$ with φ continuous and f measurable.

Suppose now that $\arctan f(x)$ is measurable. This means that

$$\{x \in X : \ \arctan f(x) > a\}$$

is measurable for all $a \in \mathbb{R}$.

If $a \geq \frac{\pi}{2}$ then $\{x \in X: \ \arctan f(x) > a\} = \varnothing$.

If $a \leq -\frac{\pi}{2}$ then $\{x \in X: \ \arctan f(x) > a\} = X$.

For $\forall a_1 \in \mathbb{R}$ there exists $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $a_1 = \tan a$. So for $a_1 \in \mathbb{R}$

$$\{x: \ f(x) > a_1\} = \{x: \ \mathrm{actan} f(x) > a\}$$

for some $a \in (-\frac{\pi}{2}, \frac{\pi}{2})$, which is measurable.

(iii). One can extend $f(x)$ to the right of the point 1 in such a way that the extension will be also differentiable. For instance, $f(1+\alpha) = f(1) + \alpha f'(1)$.

Put then $\phi_n(x) = n[f(x + \frac{1}{n}) - f(x)]$.

Then $f'(x) = \lim_n \phi_n(x)$, so f' is a limit of a sequence of measurable functions.

2 Question 2.

(a)

(i). A normed space Xb is a vector space with a norm defined on it. A norm on X is a real-valued function $\|x\|$ satisfying the following properties

(1) $\|x\| \geq 0, \ \|x\| = 0 \Longleftrightarrow x = 0;$

(2) $\|\alpha x\| = |\alpha|\|x\|$ for any scalar α and $x \in X$;

(3) $\|x + y\| \leq \|x\| + \|y\|$.

A Banach space is a normed space which is complete.

(ii). Let (x_n) be a sequence such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Denote $\sigma_k = \sum_{n=1}^{k} \|x_n\|$. Then (σ_k) is a Cauchy sequence of positive numbers. Put

$$s_k = \sum_{n=1}^{k} x_n.$$

Take $m > k$. Then $s_m - s_k = \sum_{n=k+1}^{m} x_n$,

$$\|s_m - s_k\| = \| \sum_{n=k+1}^{m} x_n\| \leq \sum_{n=k+1}^{m} \|x_n\| = \sigma_m - \sigma_k \to \text{ as } m, k \to \infty.$$

So (s_k) is a Cauchy sequence in a Banach space X. Thus (s_k) has a limit, hence the series converges.

(b)

(i) $T: \ X \mapsto Y$ is a linear operator. Then by definition

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}.$$

(ii) Let (T_n) be a Cauchy sequence in $B(X, Y)$. Thus $\forall \epsilon > 0 \ \exists N$ such that $\forall n, m > N$

$$\|T_n - T_m\| < \epsilon.$$

For all $x \in X, \ m, n > N$

$$\|T_n x - T_m x\| \leq \|T_n - T_m\|\|x\| < \epsilon\|x\|.$$

So $(T_n x)$ is a Cauchy sequence in Y. Hence $\exists y \in Y$ such that $T_n x \to y$. Define the operator $T: \ X \mapsto Y$ by $Tx := y$.

T is a linear operator.

Indeed,

$$T(\alpha x + \beta z) = \lim_n T_n(\alpha x + \beta z) = \alpha \lim_n T_n x + \beta \lim_n T_n z = \alpha T x + \beta T z.$$

$\forall x \in X, \ n > N$ we have

$$\|T_n x - Tx\| = \|T_n x - \lim_M T_m x\| = \lim_m \|T_n x - T_m x\| \leq \epsilon \|x\|.$$

Therefore $T_n = T$ is bounded, and so is $T = T_n - (T_n - T)$.

Moreover,

$$\forall n > N \quad \frac{\|T_n x - Tx\|}{\|x\|} < \epsilon \ \text{ or } \|T_n - T\| < \epsilon,$$

which proves that $\|T_n - T\| \to 0$ as $n \to \infty$.

(iii) The dual space X' to a normed space X is the space of all bounded linear functionals on X. It coincides with $B(X, \mathbb{C})$. The previous section shows that this space is a Banach space.

(c)

(i) A sequence (x_n) in a normed space X is called weakly convergent if there is an $x \in X$ such that for every $f \in X'$

$$\lim_{n \to \infty} f(x_n) = f(x).$$

(ii) If (x_n) is weakly convergent then $(f(x_n))$ is a convergent sequence of complex (or real) numbers, so it is bounded. So $\exists C_f$ such that $\forall n \in \mathbb{N}$

$$|f(x_n)| \leq C_f.$$

One can use the canonical mapping to define a sequence $(g_n) \subset X''$ by

$$g_n(f) = f(x_n).$$

Then for all $n \in \mathbb{N}$

$$|g_n(f)| = |f(x_n)| \leq C_f.$$

So the sequence $(|g_n(f)|)$ is bounded for every $f \in''$. X'' is complete. It follows from the uniform boundedness theorem that $(\|g_n\|)$ is bounded. But $\|g_n\| = \|x\|$, so the statement is proved.

(iii) Consider an orthonormal basis (e_n) in a separable Hilbert space H. Then $\|e_n - e_m\| = \sqrt{2}$ if $m \neq n$. At the same time $\forall x \in H$

$$\|x\| = \sum_{j=1}^{\infty} |\langle x, e_j \rangle|^2.$$

so that $\langle x, e_n \rangle \to 0$ $(n \to \infty)$ $\forall x \in H$. That is, $e_n \to 0$ weakly in H.

3 Question 3

(a)

(i) An inner product space is a vector space X with an inner product defined on it.

An inner product is a mapping from $X \times X$ into the scalar field K that satisfies the following properties

1) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$;
2) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$;
3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$;
4) $\langle x, x\rangle \geq 0 \ \forall x \in X$
   $\langle x, x\rangle = 0 \Longleftrightarrow x = 0$.

Examples.

1) l^2 is a space of sequences $x = (\xi_j)_{j=1}^{\infty}$ such that $\|x\| := (\sum_{j=1}^{\infty} |\xi_j|^2)^{1/2}$ is finite. It is a Hilbert space.
2) $L^2(0,1)$ is a space of Lebesgue measurable functions (more precisely equivalence classes) such that

$$\|x\| := (\int_0^1 |x(t)|^2 dt)^{1/2} < \infty.$$

   It is also a Hilbert space.

   One can introduce the same norm in the space $C[0,1]$ of continuous functions and define by this way an inner product space. This space is not complete, therefore is not a Hilbert space.

(ii) Let X be an inner product space. Then $\|x\| = \sqrt{\langle x, x\rangle}$. For any $x, y \in X$

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Proof.

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &\quad +\langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\ &= 2(\|x\|^2 + \|y\|^2).\end{aligned}$$

(iii) One cannot define the norm in l^1 by means of an inner product since the "parallelogramm equality" does not hold. To see this take

$$x = (1, 1, 0, 0, \dots),\ \|x\| = 2,$$
$$y = (1, -1, 0, 0, \dots),\ \|y\| = 2.$$

Then

$$x + y = (2, 0, 0, 0, \dots),\ \|x + y\| = 2,$$
$$x - y = (0, 2, 0, 0, \dots),\ \|x - y\| = 2.$$

(b)

(i) Let (e_k) be an orthonormal sequence in an inner product space X. The for every $x \in X$

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2.$$

Proof. Take

$$y = \sum_{k=1}^{n} \langle x, e_k\rangle e_k$$

Then

$$\|y\|^2 = \langle \sum_{k=1}^{n} \langle x, e_k\rangle e_k, \sum_{m=1}^{n} \langle x, e_m\rangle e_m\rangle = \sum_{k=1}^{n} |\langle x, e_k\rangle|^2.$$

Let $z = x - y$.Then

$$\langle z, y\rangle = \langle x, y\rangle - \langle y, y\rangle = \langle x, \sum_{k=1}^{n} \langle x, e_k\rangle e_k\rangle - \|y\|^2 = \sum_{k=1}^{n} |\langle x, e_k\rangle|^2 - \|y\|^2 = 0.$$

So that $z \perp y$. Therefore $\|x\|^2 = \|y\|^2 + \|z\|^2 \geq \|y\|^2$. We arrive at

$$\sum_{k=1}^{n} |\langle x, e_k\rangle|^2 \leq \|x\|^2$$

Passing to the limit $n \to \infty$ we obtain Bessel's inequality.

(ii). Let us calculate the norm of Tx by Parseval's identity:

$$\|Tx\|^2 = \sum_{k=1}^{\infty} 2^{-2k} |\langle x, e_{k+1}\rangle|^2 \le 2^{-2} \sum_{k=1}^{\infty} |\langle x, e_{k+1}\rangle|^2 \le \frac{1}{4}\|x\|^2.$$

So we have proved that T is bounded and that its norm $\|T\| \le \frac{1}{2}$.

Taking $x = (0, 1, 0, 0, ...)$ we have

$$\|Tx\| = \|\frac{1}{2}e_1\| = \frac{1}{2}.$$

Therefore the norm of T is $\frac{1}{2}$.

4 Question 4

(a)

1) Let X be a real vector space and p a sublinear functional on X. Let f be a linear functional defined on a subspace $Z \subset X$ such that

$$f(x) \leq p(x) \quad \forall x \in Z.$$

Then there exists an extension $\tilde{f}$ of f defined on X and such that $\tilde{f}$ is a linear functional and $\tilde{f}(x) \leq p(x)$ $\forall x \in X$.

2) Let X be a real or complex vector space and p a real-valued subaditive and homogenous functional on X.
Let f be a linear functional defined on a subspace $Z \subset X$ and such that

$$|f(x)| \leq p(x) \quad \forall x \in Z$$

. Then f has a linear extension $\tilde{f}$ from Z to X, such that

$$|\tilde{f}(x)| \leq p(x) \quad \forall x \in X.$$

(b)

(i). Suppose for a contradiction that $\delta = 0$. Then there exists a sequence $y_n \in Y$ such that

$$\|y_n - x_o\| \to 0 \text{ as } n \to \infty.$$

So x_o belongs to the closure of Y and therefore $x_o \in Y$.

(ii). Define $Z =$ span $\{Y, x_o\}$. For $z \in Z \quad z = y + t\ x_o,\ y \in Y, t$ is a scalar, and the respresentation is unique.
Let us define f on Z by

$$f(z) = t$$

If $z \in Y$ then $t = 0$, and $f(z) = 0$.

If $z = x_o$ then $t = 1$ and $f(x_o) = 1$.

Moreover,

$$|f(z)| = |t| = \frac{|t|\|z\|}{\|z\|} = \frac{\|z\|}{\|\frac{y}{t} + x_o\|} \leq \frac{\|z\|}{\delta}$$

since $\|\frac{y}{t}+x_o\| = \|x_o - (-\frac{y}{t})\| \geq 0$, where $(-\frac{y}{t}) = -\frac{1}{t}y \in Y$. So $\|f\| \leq \frac{1}{\delta}$

To prove that $\|f\| = \frac{1}{\delta}$ take a sequence $(y_n) \subset Y$ such that $\delta = \lim_n \|x_o - y_n\|$. Then $1 = f(x_o - y_n) \leq \|x_o - y_n\|\|f\| \to \delta\|f\|$. Therefore $\|f\| \geq \frac{1}{\delta}$. This finishes the proof.

(c)

(i). Linearity follows from the linearity of the integral.

$$|f(x)| = \left|\int_{-1}^{1} tx(t)dt\right| \leq \max_{[-1,1]}|x(t)| \int_{-1}^{1} |t|dt = \|x\|.$$

So that $\|f\| \leq 1$.

(ii). Let us consider the sequence of functions

$$x_n(t) = \begin{cases} 1 & \text{if } t \geq \frac{1}{n} \\ nt & \text{if } -\frac{1}{n} \leq t \leq \frac{1}{n} \\ -1 & \text{if } t \leq -\frac{1}{n} \end{cases}$$

This is an approximation (pointwise) of the function

$$\text{sgn}t = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$

Notice that $\int_{-1}^{1} t \text{ sgn}t \; d_t = 1$. Hence

$$\left|\int_{-1}^{1} t \text{ sgn}t \; d_t - \int_{-1}^{1} x_n(t)t \; dt\right| \leq \int_{-1}^{1} |t| \; |\text{sgn } t - x_n(t)|dt$$
$$= 2\int_{0}^{\frac{1}{n}} t(1-nt)dt \leq \frac{1}{n^2}.$$

Therefore

$$1 = \int_{-1}^{1} t \text{ sgn}tdt \leq \int_{-1}^{1} t \; x_n(t)dt + \frac{1}{n^2}.$$
$$f(x_n) + \frac{1}{n^2} \leq \|f\|\|x_n\| + \frac{1}{n^2} = \|f\| + \frac{1}{n^2}.$$

Passing to the limit one obtains $\|f\| \geq 1$.

5 Question 5

(a)

Since any outer measure is semi-additive, it suffices to prove that for any $E \subset X$ one has

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c). \tag{1}$$

Since $\mathfrak{A} \subset \tilde{\mathfrak{A}}$, one has

$$\mu^*(E \cap A_\varepsilon) + \mu^*(E \cap A_\varepsilon^c) \leqslant \mu^*(E). \tag{2}$$

It follows from the result of the previous exercise (for the outer measure) and from the monotonicity of μ^* that

$$|\mu^*(E \cap A_\varepsilon) - \mu^*(E \cap A)| \leqslant \mu^*((E \cap A_\varepsilon) \triangle (E \cap A)) \leqslant \mu(A_\varepsilon \cap A) < \varepsilon.$$

Therefore, $\mu^*(E \cap A_\varepsilon) > \mu^*(E \cap A) - \varepsilon$. Analogously, $\mu^*(E \cap A_\varepsilon^c) > \mu^*(E \cap A^c) - \varepsilon$.

Using (2), one obtains

$$\mu^*(E) > \mu^*(E \cap A) + \mu^*(E \cap A^c) - 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (1) follows.

(b)

Notice that

$$|f - f_n|^p \leq (|f| + |f_n|)^p \leq 2^{p-1}(|f|^p + |f_n|^p) \quad (p \geq 1)$$

Let us introduce the sequence (φ_n) by

$$\varphi_n = 2^{p-1}(|f|^p + |f_n|^p) - |f - f_n|^p$$

Then $\varphi_n \geq 0$ and $\varphi_n \to 2^p|f|^p$ a.e.
Using Fatou's lemma we have

$$\begin{aligned} 2^p\|f\|_{L^p}^p &= \|2^p|f|^p\|_{L^1} \leq \liminf_{n\to\infty} \|\varphi_n\|_{L^1} = \\ &= \liminf_{n\to\infty}\left[\int 2^{p-1}(|f|^p + |f_n|^p)d\mu - \int |f - f_n|^p d\mu\right] = \\ &= 2^p\|f\|_{L^p}^p - \limsup_{n\to\infty}\|f - f_n\|_{L^p}^p. \end{aligned}$$

Therefore $\limsup\limits_{n\to\infty}\|f - f_n\|_{L^p}^p \leq 0$.
As a result $\lim\limits_{n\to\infty}\|f - f_n\|_{L^p} = 0$.

Question 6.

(a)

The set of compact operators is a linear subspace of $B(\mathcal{H})$. Let us show the completeness. Let (T_n) be a Cauchy sequence of compact operaors. Then since $B(\mathcal{H})$ is complete there exists an operator T such that $\|T_n - T\| \to 0$ as $n \to \infty$.
All we have to prove is that T is compact itself.
Let (X_m) be a bounded sequence in $\mathcal{H}$. We have to show that (Tx_m) has a convergent subsequence.

Since T_1 is compact, (x_m) has a subsequence $(x_{1,m})$ such that $(T_1\ x_{1,m})$ is convergent. In turn $(x_{1,m})$ has a subsequence $(x_{2,m})$ such that $(T_2\ x_{2,m})$ is convergent.
Continuing in the same way we see that $(y_m) = (X_{m,m})$ is a subsequence of (X_m) such that for any fixed n $(T_n\ y_m)$ is convergent.
(x_m) is bounded, so is (y_m), $\|y_m\| \leq\ c\ \forall m$.
Let $\varepsilon > 0$. There exists N such that $\|T - T_N\| < \frac{\varepsilon}{3c}$ and such that $\|T_N\ y_k - T_N\ y_j\| < \frac{\varepsilon}{3}$ for $k, j > N$.
Hence for $k, j > N$

$$\begin{aligned}
&\|Ty_k - Ty_j\| \leq \|Ty_k - T_N yk + T_N y_k - T_N y_j + T_N y_j - Ty_j\| \leq \\
&\leq\ \|Ty_k - T_N y_k\| + \|T_N y_k - T_N y_j\| + \|T_N y_j - Ty_j\| \leq \\
&\leq \|T - T_N\|\ \|y_k\| + \|T_N y_k - T_N y_j\| + \|T_N - T\|\ \|y_j\| < \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

So T is compact.

(b)

The distance between $\mathcal{K}(\mathcal{H})$ and Id is defined by

$$d = \inf_{A \in \mathcal{K}(\mathcal{H})} \|Id - A\|$$

Zero-operator is obviously compact. So

$$d \leq \|Id\| = 1.$$

Suppose now that $d < 1$. Then there exists a compact operator A such that

$$\|Id - A\| < 1 - \varepsilon.$$

This means that $A = Id - (Id - A)$ is invertible and the inverse is bounded. But A is compact, therefore $\mathcal{H}$ is finite dimensional which contradicts the conditions.

Question 7.

(a),(b)

Let (e_j), (φ_j) be bases in $\mathcal{H}$ and we are given that

$$\sum_{j=1}^{\infty} \|A\ e_j\|^2 \ < \ \infty.$$

Then

$$\begin{aligned}
&\sum_{j=1}^{\infty} \|A\ \varphi_j\|^2 \ = \ \sum_k \sum_j |\langle A\varphi_j, e_k\rangle|^2 = \\
&= \sum_j \sum_k |\langle \varphi_j,\ A^*\ e_k\rangle|^2 = \sum_k \|A^*\ e_k\|^2 = \\
&= \sum_j \sum_k |\langle e_j,\ A^*\ e_k\rangle|^2 = \sum_k \sum_j |\langle\ Ae_j, e_k\rangle|^2 = \\
&= \sum_j \|A\ e_j\|^2 < \infty.
\end{aligned}$$

(c)

Let $x \in \mathcal{H}$, $\|x\| = 1$. Then $x = \sum_{j=1}^{\infty} \alpha_j e_j$, $\sum_{j=1}^{\infty} |\alpha_j|^2 = 1$. Then

$$\|Ax\|^2 = \sum_j |\langle Ax,\ e_j\rangle|^2 = \sum_j |\sum_k \alpha_k \langle Ae_k, e_j\rangle| \leq$$

(Swartz's inequality)

$$\begin{aligned}
&\leq \sum_j \left(\sum_k |\alpha_k|^2 \ \sum_k |\langle Ae_k, e_j\rangle|^2 \right) = \\
&= \sum_j \sum_k |\langle Ae_k, e_j\rangle|^2 = \sum_k \|Ae_k\|^2 = \|A\|_2.
\end{aligned}$$

So $\|A\| = \sup_{\|x\|=1} \|A\ X\| \leq \|A\|_2$.

(d)

Let $A, B \in \sigma_2(\mathcal{H}), \alpha, \beta \in \mathbb{C}$. Then

$$\sum_n \|(\alpha\ A + \beta\ B)e_n\|^2 \leq \sum_n (|\alpha|\ \|Ae_k\|\ +\ |\beta|\ \|Be_n\|)^2 \leq$$
$$\leq\ 2\left(|\alpha|^2 \sum_n \|Ae_n\|^2 + |\beta|^2\ \sum \|Be_n\|^2\right)\ <\ \infty.$$

So $\sigma_2(\mathcal{H})$ is a linear subspace of $B(\mathcal{H})$.
The above inequality also gives

1) $\|\alpha\ A\|_2 = |\alpha|\ \|A\|$

2) $\|A + B\|_2 \leq \|A\|_2 + \|B\|_2$

Assume now that $\|A\|_2 = 0$. Then $Ae_j = 0\ \forall_j$
Then for any $x \in \mathcal{H},\ x = \sum \alpha_j e_j,\ Ax = \sum \alpha_j Ae_j = 0$.
So $A = 0$.

Therefore $\|A\|_2$ is a norm on $\sigma_2(\mathcal{H})$.

(e)

$(A, B) = \sum_{j=1}^{\infty} \langle Ae_j,\ Be_j\rangle$ Then

1) $(A + C, B) = \sum_j \langle (A + C)e_j,\ Be_j\rangle\ =\ \sum_j\ \langle Ae_j,\ Be_j\rangle$
$+ \sum_j \langle Ce_j,\ Be_j\rangle$

2) $(\alpha A, B) = \sum_j \langle \alpha Ae_j,\ Be_j\rangle\ =\ \alpha \sum_j \langle Ae_j,\ Be_j\rangle$

3) $\overline{(B, A)} = \sum_j \overline{\langle Be_j,\ Ae_j\rangle} = \sum_j \langle Ae_j,\ Be_j\rangle = (A, B)$

4) $(A, A) = \sum_j \|Ae_j\|^2$

So (A, B) defines an inner product in $\mathcal{H}$.

(f)

The only fact we have to prove is completeness of $\sigma_2(\mathcal{H})$. Let (A_n) be a Cauchy sequence in $\sigma_2(\mathcal{H})$, i.e. $\|A_n - A_m\|_2 \to 0$ as $m, n \to \infty$.
Then $\|A_n - A_m\| \to 0$ by (c).
So there is an $A \in \sigma_2(\mathcal{H})$ such that $\|A_n - A\| \to 0$.
Let $\varepsilon > 0$. There is such N that $\|A_n - A_m\|_2 < \varepsilon$ for $m, n > N$.
Then $\sum_{k=1}^{\ell} \|(A_n - A_m)e_k\|^2 \le \|A_n - A_m\|_2^2 < \varepsilon^2$, for $m, n > N$ and any ℓ. Passing to the limit $m \to \infty$ we obtain

$$\sum_{k=1}^{\ell} \|(A_n - A)e_k\|^2 \le \varepsilon^2$$

for $n > N$. Therefore $\|A_n - A\|_2 \le \varepsilon \ \forall n > N$.
This implies that $A_n - A \in \sigma_2(\mathcal{H})$, $\Rightarrow A \in \sigma_2(\mathcal{H})$ and

$$\|A_n - A\|_2 \to 0 \quad \text{as} \quad n \to \infty.$$

(g)

$$\|A\ B\|_2^2 = \sum_{kj} |\ \langle\ ABe_k, e_j\rangle\ |^2 = \sum_{kj} |\ \langle e_k, B^* A^* e_j\rangle|^2 =$$
$$= \sum_j \|B^* A^* e_j\|^2 \le \|B^*\|\ \|A\|_2 - \|B\|\ \|A\|_2$$
$$\|B\ A\|_2^2 = \sum_j \|BAe_j\|^2 \le \|B\| \sum_j \|Ae_j\|^2 = \|B\|\ \|A\|_2.$$