Solutions to Measure Theory and Functional Analysis 1997/98 exam

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Question 1.

(a)

Let $\mathfrak{A}$ be an algebra of sets. A function $\mu : \mathfrak{A} \mapsto \mathbb{R}^+ \cup \{\infty\}$ is called a measure if

1) $\mu(\emptyset) = 0$
2) for any disjoint family of sets $(A_i)_{i=1}^{\infty}$ from $\mathfrak{A}$ such that

$$\bigsqcup_{i=1}^{\infty} A_i \in \mathfrak{A}$$

one has

$$\mu(\bigsqcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

(b)

(i) Let $\mathfrak{A}$ be a σ-algebra of sets, μ be a measure defined on it.
Let $(A_i)_{i=1}^{\infty} \subset \mathfrak{A}$ be a monotonically increasing sequence of sets. Then

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(A_n).$$

Analogous property holds for a monotonically decreasing sequence. Namesly, if $\mu(A_i) < \infty$ then

$$\mu(\bigcap_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \mu(A_n).$$

Proof: Notice that

$$\bigcup_{i=1}^{\infty} A_i = A_1 \sqcup \bigsqcup_{i=2}^{\infty} (A_i \setminus A_{i-1})$$

Using the property of subtractivity and σ-additivity of a measure we get

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \mu(A_n) + \sum_{i=2}^{\infty} (A_i \setminus A_{i-1}) =$$

$$= \mu(A_1) + \lim_{n\to\infty} \sum_{i=2}^{\infty} \mu(A_i) - \mu(A_{i-1})) =$$

$$= \lim_{n\to\infty} (\mu(A_n).$$

Let now (A_i) be monotonically decreasing.
Then

$$A_1 \setminus \bigcap_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty}(A_1 \setminus A_i).$$

Therefore

$$\bigcap_{i=1}^{\infty} A_i = A_1 \setminus \bigcup_{i=1}^{\infty}(A_1 \setminus A_i)$$

and

$$\mu(\bigcap_{i=1}^{\infty} A_i) = \mu(A_1) - \mu(\bigcup_{i=1}^{\infty}(A_1 \setminus A_i))$$

$(A_1 \setminus A_i)_{i=1}^{\infty}$ is an increasing sequence. Hence

$$\begin{aligned}\mu(\bigcap_{i=1}^{\infty} A_i) &= \mu(A_1) - \lim_{n\to\infty} \mu(A_1 \setminus A_n)\\ &= \mu(A_1) - \mu(A_1) + \lim_{n\to\infty} \mu(A_n).\end{aligned}$$

(ii) Let x_n be a decreasing sequence, $x_n \to x$. One has

$$A \cap (-\infty, x] = \bigcap_n (A \cap (-\infty, x_n])$$

Consequently by (i)

$$\mathrm{m}eas(A \cap (-\infty, x]) = \lim_n \mathrm{m}eas(A \cap (-\infty, x])$$

Let x_n be a strictly increasing sequence, $x_n \to x$. Then

$$A \cap (-\infty, x) = \bigcup_n (A \cap (-\infty, x_n])$$

Therefore, since $\mathrm{m}eas(\{x\}) = 0$,

$$\mathrm{meas}(A \cap (-\infty, x)) = \lim_n \ \mathrm{meas}(A \cap (-\infty, x]).$$

(c)

Since $A \subset (A \triangle B) \cup B$, by the monotonicity and semi-additivity of a measure one has

$$\mu(A) \le \mu(A \triangle B) + \mu(B)$$

Analogously,$\mu(B) \le \mu(A \triangle B) + \mu(B)$.

(d)

Since $\mu(A_i^c) = 1 - \mu(A_i)$ we have

$$\sum_{i=1}^{n} \mu(A_i^c) = n - \sum_{i=1}^{n} \mu(A_i) < 1.$$

By the semi-additivity of a measure

$$\mu(\bigcup_{i=1}^{n} A_i^c) \leq \sum_{i=1}^{n} \mu(A_i^c) < 1.$$

Therefore

$$\mu(\bigcap_{i=1}^{n} A_i) = 1 - \mu((\bigcap_{i=1}^{n} A_i)^c) =$$

$$= 1 - \mu(\bigcup_{i=1}^{n} A_i^c) > 0.$$

Question 2.

(a)

Let $\mathfrak{A}$ be a σ-algebra of subsets of X, μ a measure on it.
The relation on the set of σ-measurable functions $(f \sim g)$ if and only if $(f(x) \sim g(x)$ for μ-almost every $x \in X$ is a relation of equivalency.
The set of classes of equivalency with the property

$$\|f\|_p \ := \left(\int_X \ |f|^p d\mu\right)^{1/p} < \infty$$

is called an L^p-space. The latter quantity is the norm in this space.

(b)

(i)
Let $(X, \mathfrak{A}, \mu)$ be a measure space.
Let $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$. Let f and g be measurable functions, $|f|^p$ and $|g|^q$ be integrable. Then fg is measurable and the following inequality holds.

$$\int_X |fg| d\mu \leq \left(\int_X |f|^p d\mu\right)^{1/p} \left(\int_X |g|^q d\mu\right)^{1/q}.$$

Let us prove first a couple of lemmas.
Lemma 1. For p and q as above, $a > 0,\ b > 0$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. Note that $x^p - px + p - 1 \geq 0$ for $x > 0$
Take $x = ab^{-\frac{1}{p-1}}$.

Lemma 2. If $p \geq 1,\ a,\ b \in \mathbb{R}$ then

$$|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p).$$

Proof. $p=1$ is obvious. For $p>1$ the function $y=x^p,\ x\geq 0$, is convex. Therefore

$$\left(\frac{|a|+|b|}{2}\right)^p \leq \frac{|a|^p+|b|^p}{2}\ .$$

Proof of Hölder's inequality .
It suffices to consider the case

$$\|f\|_p = \left(\int_X |f|^p d\mu\right)^{1/p} > 0, \|g\|_q > 0.$$

Let $\quad a=\dfrac{|f(x)|}{\|f\|_p},\ b=\dfrac{|g(x)|}{\|g\|_q}$

By Lemma 1

$$\frac{|f(x)g(x)|}{\|f\|_p\|g\|_q} \leq \frac{|f|^p}{p\|f\|_p^p}+\frac{|g|^q}{q\|g\|_q^q}$$

After integration

$$\frac{\int_X |fg|d\mu}{\|f\|_p\|g\|_q} \leq \frac{1}{p}+\frac{1}{q}=1.$$

(ii) It suffices to prove that $\displaystyle\int_X |f|^q d\mu < \infty \quad$ if $\qquad f\in L^p\cap L^r$.
By Hölder's inequality

$$\int_X |f|^q d\mu = \int_X |f|^{q(\frac{p}{q}\cdot\frac{r-q}{r-p}+\frac{r}{q}\cdot\frac{q-p}{r-p})} d\mu \leq$$

$$\leq \|f\|_p^{\frac{p}{q}\cdot\frac{r-q}{r-p}}\ \|f\|_r^{\frac{r}{q}\cdot\frac{q-p}{r-p}}$$

which proves the statement.

(iii) Let $\varphi(p)=p\log\|f\|_p$ for a fixed $f\in\cap_{p\geq 1}L^p$. By definition of convexity one has to show that $\varphi(\alpha p+\beta q)\leq \alpha\varphi(p)+\beta\varphi(q)$ for $\alpha+\beta=1$ and $p,q\geq 1$. Since $\varphi(p)=\log\|f\|_p^p$ the desired inequality can be rewritten in the form

$$\log\|f\|_{\alpha p+\beta q}^{\alpha p+\beta q} \leq \log\|f\|_p^{\alpha p}+\log\|f\|_q^{\beta q} \quad (*)$$

One has by Hölder's inequality

$$\|f\|_{\alpha p+\beta q}^{\alpha p+\beta q} = \int_X |f|^{\alpha p+\beta q} d\mu \leq$$

$$\leq \left(\int_X \left(|f|^{\alpha p}\right)^{\frac{1}{\alpha}} d\mu\right)^{\alpha} \left(\int_X \left(|f|^{\beta q}\right)^{\frac{1}{\beta}} d\mu\right)^{\beta} =$$

$$= \|f\|_p^{\alpha p}\|f\|_q^{\beta q}.$$

Taking logarithm we obtain the inequality (*).

(c)

Take a representative $\tilde{f} \in f$ which is a measurable function defined a.e. on the ball. Then the function

$$e^{\alpha|\tilde{f}|} = \sum_{k=0}^{\infty} \frac{\alpha^k |\tilde{f}|^k}{k!}$$

is measurable as a.e. limit of measurable function. One has

$$\int_{B_1} e^{\alpha|f|} dx \;:= \int_{B_1} e^{\alpha|\tilde{f}|} dx =$$

$$\int_{B_1} dx + \sum_{k=1}^{\infty} \int_{B_1} \frac{\alpha^k |\tilde{f}|^k}{k!} dx =$$

$$= m(B_1) + \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} \|f\|_k^k \; .$$

It remains to show that the series $\sum_{k=1}^{\infty} \dfrac{\alpha^k}{k!} \|f\|_k^k$

is convergent. Note that $a_k = \dfrac{\alpha^k}{k!} \|f\|_k^k$. Then $\dfrac{a_{k+1}}{a_k} = \dfrac{\alpha}{k+1} \cdot \dfrac{\|f\|_{k+1}^{k+1}}{\|f\|_k^k}$.

Using the inequality $\|f\|_{k+1} \le k_k^{\frac{1}{k+1}}$

one has $\quad \|f\|_{k+1}^{k+1} \le k\|f\|_k^k$, $\quad$ so that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \alpha \lim_{k\to\infty} \frac{k}{k-1} = \alpha.$$

By the ratio test the series is convergent if $0 < \alpha < 1$.

Question 3.

(a)

(i) A linear operator from one normed space X to another normed space Y is a mapping $T : \mathcal{D}(X) \to Y$ satisfying the condition $\mathcal{D}(T)$ in a subspace of X.

$$T(\alpha x + \beta y) = \alpha T x + \beta T y \;\forall x, y \in \mathcal{D}(T)$$

and $\alpha, \beta \in K$, where $K = \mathbb{R}$ or $K = \mathbb{C}$ depending on whether X and Y are defined over real or comples field of scalars.
A linear operator T is called bounded if there exists a constant M such that for all $x \in \mathcal{D}(T)$

$$\|Tx\|_Y \le M\|x\|_X$$

(ii) The infimum of the constants M in the above inequality is called the norm of the operator T.

(b)

(i) For $T = 0$ the statement is trivial. Let $T \neq 0$. Then $\|T\| \neq 0$. Assume T is bounded, $x_0 \in \mathcal{D}(T)$. For any $\epsilon > 0$ and any $x \in \mathcal{D}(T)$ such that

$$\|x - x_0\| < \delta, \ \delta = \frac{\epsilon}{\|T\|}$$

we obtain

$$\|Tx - Tx_0\| \leq \|T\| \, \|x - x_0\| < \epsilon.$$

Hence T is continuous.
Conversely, let T be continuous at an arbitrary point $x_0 \in \mathcal{D}(T)$. Then for any $\epsilon > 0$ there is a $\delta > 0$ such that for all $x \in \mathcal{D}(T)$ satisfying $\|x - x_0\| \leq \delta$ we have

$$\|T_x - Tx_0\| \leq \epsilon$$

Take any $y \neq 0, \ y \in \mathcal{D}(T)$, set $x = x_0 + \frac{\delta}{\|x\|} y$
Then $x - x_0 = \frac{\delta}{\|x\|} y$. $\|x - x_0\| = \delta$ and $\|Tx - Tx_0\| = \frac{\delta}{\|y\|} \|Ty\|$. Then we get

$$\|Ty\| \leq \frac{\epsilon}{\delta} \|y\|$$

which shows that T is bounded.

(ii) Continuity of T at a point implies that T is bounded by the second part of the proof of (i), which implies continuity by (i).

(c)

A is linear because of the linearity of an integral. Let us estimate the value $\|Ax\|$:

$$\|Ax\| = \sup_{t \in [0,1]} | \int_0^t x(\tau) d\tau | \leq$$

$$\leq \sup_{\tau \in [0,1]} |x(\tau)| \sup_{\tau \in [0,1]} \int_0^t d\tau = \|x\|$$

therefore A is bounded with norm less than or equal to one.
Take $x(t) = 1$ on $[0, 1]$. Then $A(x)(t) = \displaystyle\int_0^t d\tau = t$ and $\|Ax\| = 1$. Therefore $\|Ax\| = 1$.

(d)

Let us define the operator e^A as the series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

Since the space of bounded linear operators on X, $B(X)$, is a Banach space, it follows that the series $\displaystyle\sum_{k=0}^{\infty} \frac{A^k}{k!}$ is uniformly convergent (in uniform operator topology; therefore it defines a linear bounded operator) if and only if it is absolutely convergent.

Hence it is enough to check the confergence of the numerical series $\sum_{k=0}^{\infty} \frac{\|A^k\|}{k!}$. The last follows from the inequality $\|A^k\| \leq \|A\|^k$.

Now $\|e^A\| \leq \sum_{k=0}^{\infty} \frac{\|A^k\|}{k!} \leq \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!} = e^{\|A\|}$.

Question 4.

(a)

An inner product space is a vector space X with an inner product defined on X, which is a mapping of $X \times X$ into the scalar field $K: \langle x, y\rangle \in K$, such that

1) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

2) $\langle \alpha x + y\rangle = \alpha\langle x, y\rangle$

3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

4) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$.

One defines the norm in X by

$$\|x\| = \langle x, x\rangle^{\frac{1}{2}}.$$

If X is complete with respect to this norm, it is called Hilbert space.

(b)

Every bounded linear functional f on a Hilbert space H can be represented in the form

$$f(x) = \langle x, z\rangle \qquad (*)$$

where z is an element of H uniquely determined by f and

$$\|f\| = \|z\|. \qquad (**)$$

Proof. If $f = 0$ then (*) holds with $z = 0$. Let $f \neq 0$. The null-set of f is a closed subspace in H, $N(f) \neq H$, since $f \neq 0$.

Therefore $N(f)^{\perp}$ contains a $z_0 \neq 0$. Set

$$v = f(x)z_0 - f(z_o)x \text{ for } x \in H.$$

Hence

$$f(v) = 0, \text{ therefore } v \in N(f).$$

one has

$$0 = \langle v, z_0\rangle = \langle f(x)z_0 - f(z_0)x, z_0\rangle =$$
$$= f(x)\langle z_0, z_0\rangle - f(z_0)\langle x, z_0\rangle$$

Hence

$$f(x) = \frac{f(z_0)}{\langle z_0, z_0\rangle}\langle x, z_0\rangle$$

Take $z = \dfrac{\overline{f(z_0)}}{\langle z_0, z_0\rangle} z_0$ to obtain (*).

If $f(x) = \langle x, z_1\rangle = \langle x, z_2\rangle$ then $\langle x, z_1 - z_2\rangle = 0$ for all x, in particular, for $x = z_1 - z_2$ and the element z in (*) is uniquely defined.

From (*) with $x = z$
$$\|z\|^2 = \langle z, z\rangle = f(z) \le \|f\| \, \|z\|$$
so $\|f\| \ge \|z\|$. On the other hand
$$|f(x)| = |\langle x, z\rangle| \le \|x\|\|z\|, \text{ and } \|f\| \ge \|z\|$$
Hence $\|f\| = \|z\|$.

(c)

(i) Let $T : H_1 \to H_2$ be a linear bounded operator, where $H_1,\ H_2$ are Hilbert spaces. Then the Hilbert-adjoint T^* of T is a linear operator
$$T^* : H_1 \to H_2$$
such that for all $x \in H_1,\ y \in H_2$
$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$
(ii) Set
$$B = \frac{1}{2}(A + A^*)$$
$$B = \frac{1}{2i}(A - A^*)$$
Then B, C are bounded linear operators.
Using properties of the Hilbert-adjoint operator we have
$$\begin{array}{rcl} B^* & = & \frac{1}{2}(A + A^{**}) = \frac{1}{2}(A^* + A) = B \\ C^* & = & \left(\frac{1}{2i}(A - A^*)\right)^* = \frac{-1}{2i}A^* + \frac{1}{2i}A^{**} = \\ & = & \frac{1}{2i}(A - A^*) = C. \end{array}$$
Assume that $A = B_1 + iC_1 = B_2 + iC_2$ where $B_1,\ B_2,\ C_1,\ C_2$ are self-adjoint. Then
$$(B_1 - B_2) + i(C_1 - C_2) = 0 \text{ and}$$
$$\langle (B_1 - B_2)x, x\rangle + i(C_1 - C_2)x, x\rangle = 0$$
Since $B_1 - B_2$ and $C_1 - C_2$ are self-adjoint, $\langle (B_1 - B_2)x, x\rangle$ and $\langle (C_1 - C_2)x, x\rangle$ are real numbers, and it follows that $\langle (B_1 - B_2)x, x\rangle = 0$ and $\langle (C_1 - C_2)x, x\rangle = 0$ for all x and y. Therefore $B_1 = B_2$ and $C_1 = C_2$ which proves the uniqueness of the representation.

(iii) By the definition of the Hilbert-adjoint for A
$$\langle Ax, y\rangle = \langle x, A^*y\rangle$$
So $z = A^*y,$ where$\langle x, z\rangle = \langle Ax, y\rangle$.
$$\langle Ax, y\rangle = \int_0^1 \int_0^1 (t - s)x(s)ds\, \overline{y(t)}dt =$$

$$= \int_0^1 \int_0^1 x(s)\overline{(t-s)y(t)}dtds =$$

$$= \int_0^1 x(s)\overline{\int_0^1 (t-s)y(t)dtds} = \int_0^1 x(s)\overline{z(s)}ds$$

where $z(s) = \int_0^1 (t-s)y(t)dt$ Changing t and s we find

$$(A^*y)(t) = \int_0^1 (s-t)y(s)ds$$

Question 5.

(a)

From the inequality $\|A^j\| \leq \|A\|^j$, $j \in \mathbb{N}$, it follows that the series $\sum_{j=0}^{\infty} A^j$ is absolutely convergent ($\|A\| < 1$). Since $(B(X))$ is complete, there exists $S \in B(X)$, such that $S = \sum_{j=0}^{\infty} A^j$. Let $S_n = \sum_{j=0}^{n} A^j$. Then

$$(I-A)S_n = S_n(I-A) = I - A^{n+1}.$$

Let $n \to \infty$. Then $A^{n+1} \to 0$ because $\|A\| < 1$.
Thus $S = (I-A)S = S(I-A) = I$.
So $S = (I-A)^{-1}$. $\|S\| \leq \sum_{j=0}^{\infty} \|A\|^j = \frac{1}{1-\|A\|}$.

(b)

(i) $B = A - (A-B) = [I - (A-B)A^{-1}]A$ since A is invertible. From the given inequality $\|(A-B)A^{-1}\| \leq \|A-B\| \, \|A^{-1}\| < 1$.
Therefore by (a) $I - (A-B)A^{-1}$ is invertible and the inverse is bounded

$$B^{-1} = A^{-1}[I - (A-B)A^{-1}]^{-1}$$

$$\|B^{-1}\| \leq \|A^{-1}\| \, \|[I - (A-B)A^{-1}]^{-1}\| \leq$$

$$\leq \frac{\|A^{-1}\|}{1 - \|A-B\| \, \|A^{-1}\|}.$$

(ii) Let us denote by $\mathcal{S}(X)$ the set of all invertible operators in $\mathcal{B}(X)$. If $A \in \mathcal{S}(X)$ with $\|A^{-1}\| = r$ then the ball in $\mathcal{B}(X)$ centered at A with radius r belongs to $\mathcal{S}(X)$ since for every B from the ball $\|B - A\| < \|A^{-1}\|$ and the statement follows from (i)

(c)

$A = I - (I - A)$ and by (a) is ivertible with bounded inverse. A is compact. Therefore the Banach space X is finite dimensional.

(d)

If $T = 0$, then $\sigma(T) = \{0\} \neq \emptyset$. Let $T \neq 0,$ then $\|T\| \neq 0$.
For $|\lambda| > \|T\|$ define

$$R_\lambda = -\frac{1}{\lambda}\sum_{j=0}^{\infty}(\frac{1}{\lambda}T)^j$$

The series is absolutely convergent for, say, $|\lambda| \geq 2\|T\|$:

$$\|R_\lambda\| \leq \frac{1}{|\lambda|}\sum_{j=0}^{\infty} \|\frac{1}{\lambda}T\|^j = \frac{1}{|\lambda| - \|T\|} \leq \frac{1}{\|T\|}$$

Suppose, for a contradiction, that $\sigma(T) = \emptyset$, then $\rho(T) = \mathbb{C}$. R_λ is holomorphic for all $\lambda \in \mathbb{C}$. Therefore it is uniformly bounded on the compact disk $|\lambda| \leq 2\|T\|$, and we have proved that it is uniformly bounded outside the disk. Hence it is uniformly bounded on the whole plane $\mathbb{C}$. Thus by Liouville's theorem R_λ is an element of $B(X)$ independent of λ, as an entire function bounded on $\mathbb{C}$. This implies that R_λ^{-1} is independent from λ, but $R_\lambda^{-1} = T - \lambda$ which is a contradiction.

Question 6.

(a)

A bounded linear operator $A : H \to H$ on a Hilbert space H is called self-adjoint if for any $x, y \in X$

$$\langle Ax, y\rangle = \langle x, Ay\rangle$$

(b)

Suppose that$A$ is a self-adjoint operator on a Hilbert space H. Let λ be its eigenvalue. Then there is an $0 \neq x \in H$ such that $Ax = \lambda x$. Using the self-adjointness of A we obtain

$$\lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Ax, x\rangle = \langle x, Ax\rangle = \langle x, \lambda x\rangle$$

$$= \overline{\lambda}\langle x, x\rangle.$$

Since $\langle x, x\rangle \neq 0$, we conclude that $\lambda = \overline{\lambda}$. Hence λ is real.
Let λ and μ be eigenvalues, x, y corresponding eigenvectors of $A,\ \lambda \neq \mu$. Then $Ax = \lambda x,\ Ay = \mu y$. λ and μ are real as proved above. From the self-adjointness of A

$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Ax, y\rangle = \langle x, Ay\rangle = \mu\langle x, y\rangle.$$

Therefore $\langle x, y\rangle = 0$

(c)

(i)

$$\|(A - \lambda)x\|^2 = \langle (A - \nu)x, (A - \nu)x\rangle$$

$$+\langle (A - \nu)x, i\mu x\rangle + \langle i\mu x, (A - \nu)x\rangle + |\mu|^2\|x\|^2$$

(due to the self-adjointness of A)

$$= \|(A - \nu)x\|^2 + |\mu|^2\|x\|^2$$

from which the inequality follows.

(ii) The inequality in (i) means that the inverse to $(A-\lambda)$ is bounded on its domain of the definition, which is $\mathrm{Ran}(A-\lambda)$. Since $(A-\lambda)$ is bounded and defined on the whole space H, it is closed, therefore the inverse is closed too.
Hence its domain is a closed subspace of H.

(iii) Suppose for a contradiction that there is an element $y \in H,\ y \neq 0$, which is orthogonal to $\mathrm{Ran}(A-\lambda)$, that is

$$\langle (A-\lambda)x, y\rangle = 0 \text{ for any } x \in H.$$

By the self-adjointness of A it follows that

$$\langle x,\ (A-\overline{\lambda})y\rangle = 0 \text{ for any } x \in H.$$

Therefore $(A-\overline{\lambda})y = 0$, so y is an eigenvector of A, corresponding to $\overline{\lambda} = \nu - i\mu$, where $\mu \neq 0$. Thus $(A-\overline{\lambda})$ is not invertible which contradicts to (i).

(iv) If $\mu \neq 0$ then by (i) and (iii) $A-\lambda$ is invertible and the inverse is bounded.
By definition of the resolvent $R_\lambda(A) = (A-\lambda)^{-1}$ so $\lambda \in \rho(A)$-resolvent set.
Denote $y = (A-\lambda)x$. Then $x = R_\lambda(A)y$. Using the inequality (i) for y we have

$$\|y\| \geq |\mu|\ \|R_\lambda(A)y\|$$

or else, $\dfrac{\|R_\lambda(A)y\|}{\|y\|} \leq \dfrac{1}{|\mathrm{Im}\lambda|}$.
Taking supremum over $y \in H$ we obtain the desired.