MELLIN TRANSFORM AND RIEMANN ZETA FUNCTION

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Definiton

Define $g(s) = \mathcal{M}\{f\}(s) = \int_0^\infty f(t)t^{s-1}\mathrm{d}t$ and f is recoverable: $f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{x+i\infty} x^{-s}g(s)\mathrm{d}x$.
Note: We need to restrict s to values where the integral will converge. We have $g(s)$ exists if $\int_0^\infty |f(x)|x^{s-1}\mathrm{d}x$ is bounded for some $k > 0$. And $f(x)$ will have $c > k$.

Examples

(1) The Γ function is defined $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\mathrm{d}t$.
Immediately, we see that $\Gamma(s) = \mathcal{M}\{f\}(s)$ where $f(t) = e^{-t}$.
Notice this analytically continues Γ to $\mathbb{C}$ except at $s = 0, -1, -2, \ldots$.
Aside: Γ has some nice properties that we will be utilizing later:

$$\Gamma(s+1) = \Gamma(s) \qquad\qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

(2) Let $f(x) = \frac{1}{e^x - 1}$. And let $\mathrm{Re}(s) > 1$.

$$\begin{aligned}
\mathcal{M}\{f\}(s) &= \int_0^\infty x^{s-1}\frac{1}{e^x-1}\mathrm{d}x \\
&= \int_0^\infty x^{s-1}\sum_{n=1}^\infty e^{-nx}\mathrm{d}x \\
&= \sum_{n=1}^\infty \int_0^\infty x^{s-1}e^{-nx}\mathrm{d}x \\
&\ \text{Set:}\quad nx = t \quad \mathrm{d}t = n\mathrm{d}x \\
&= \sum_{n=1}^\infty \int_0^\infty \left(\frac{t}{n}\right)^{s-1} e^{-t}\frac{1}{n}\mathrm{d}t \\
&= \sum_{n=1}^\infty n^{-s}\int_0^\infty t^{s-1}e^{-t}\mathrm{d}t \\
&= \zeta(s)\Gamma(s).
\end{aligned}$$

Properties

Note: There are many interesting properties of the Mellin transform, but we will only cover a few of interest.

(1) $\mathcal{M}\{f(ax)\}(s) = \int_0^\infty f(ax)x^{s-1}\mathrm{d}x = a^{-s}\int_0^\infty \mathcal{M}\{f\}(s)$

Example: Let $f(t) = e^{-t}, c > 0, f(ct) = e^{-ct}$.

We already know that $\mathcal{M}\{f\}(s) = \Gamma(s)$. So, we anticipate $\mathcal{M}\{f(ct)\}(s) = c^{-s}\Gamma(s)$.

$$\begin{aligned}
\mathcal{M}\{e^{-ct}\}(s) &= \int_0^\infty e^{-ct}t^{s-1}\frac{\mathrm{d}t}{t} \\
& \quad u = ct \Rightarrow \mathrm{d}u = c\mathrm{d}t \\
&= \int_0^\infty e^{-u}\left(\frac{u}{c}\right)^s\left(\frac{1}{c}\right)\left(\frac{c}{u}\right)\mathrm{d}u \\
&= c^{-s}\int_0^\infty e^{-u}u^{s-1}\mathrm{d}u \\
&= c^{-s}\Gamma(s).
\end{aligned}$$

(2) $\mathcal{M}\{x^a f(x)\}(s) = \int_0^\infty f(x)x^a x^{s-1}\mathrm{d}x = \int_0^\infty f(x)x^{(a+s)-1}\mathrm{d}x = \mathcal{M}\{f(x)\}(s+a)$.

(3) $\mathcal{M}\left\{\frac{1}{x}f\left(\frac{1}{x}\right)\right\}(s) = \mathcal{M}\{f(x)\}(1-s)$.

Recall

The theta function: $\theta(t) = \sum_{-\infty}^{\infty} e^{-\pi t n^2}$ $(\mathrm{Re}(t) > 0)$ has properties:

(1) $\theta(t) = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right)$. The proof will use Poisson and Fourier transforms.

(2) $\left|\theta(t) - t^{-\frac{1}{2}}\right| < e^{\frac{-c}{t}} : c > 0 \Rightarrow \theta(t) \ll t^{\frac{-1}{2}}$.

Idea:
We are going to relate θ and ζ. In particular, there is a relationship between the Riemann Zeta function and the Mellin transform of the Theta function. This relationship (along with the functional equation for the Theta function) will provide insight on the Riemann Zeta function. In particular, we will get a functional equation and meromorphic continuation for the Riemann Zeta function.

Theorem

Let $\zeta(s) = \sum_{n=1}^{\infty}$ for $\mathrm{Re}(s) > 1$. We have that ζ extends analytically onto $\mathbb{C}$ except for a simple pole at $s = 1$ with Residue of 1. And if we define $\Lambda(s) = \pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$, then we obtain the functional equation: $\Lambda(s) = \Lambda(s-1)$. In particular, $\zeta(s)$ satisfies the functional equation

$$\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{-(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Proof

Define

$$\phi(s) = \int_1^{\infty} t^{\frac{s}{2}}(\theta(t)-1)\frac{\mathrm{d}t}{t} + \int_0^1 t^{\frac{s}{2}}\left(\theta(t) - \frac{1}{\sqrt{t}}\right)\frac{\mathrm{d}t}{t}.$$

Note that $\theta(t) - 1 = 2\sum_{n=1}^{\infty} e^{-\pi n^2 t} \to 0$ at $\infty \Rightarrow$ the integral converges.
Also, since $\theta(t) \ll t^{\frac{-1}{2}}$, the second integral will converge, as well.
Let $\mathrm{Re}(s) > 1$ and focus on the second integral:

$$\int_0^1 t^{\frac{s}{2}}\theta(t)\frac{\mathrm{d}t}{t} - \int_0^1 t^{\frac{s-1}{2}}\frac{\mathrm{d}t}{t} = \int_0^1 t^{\frac{s}{2}}\theta(t)\frac{\mathrm{d}t}{t} - \frac{2}{s-1}.$$

$$\begin{aligned}
\phi(s) &= 2\int_1^\infty \sum_{n=1}^\infty e^{-\pi n^t} t^{\frac{s}{2}} \frac{\mathrm{d}t}{t} + \int_0^1 t^{\frac{s}{2}} \frac{\mathrm{d}t}{t} + 2\sum_{n-1}^\infty \int_0^1 t^{\frac{s}{2}} e^{-\pi n^2 t} \frac{\mathrm{d}t}{t} - \frac{2}{s-1} \\
&= 2\sum_{n=1} \int_0^\infty e^{-\pi n^2 t} t^{\frac{s}{2}} \frac{\mathrm{d}t}{t} + \frac{2}{s} + \frac{2}{1-s}
\end{aligned}$$

Letting $c \to \pi n^2, s \to \dfrac{s}{2}$, and using the Scaling property previously discussed:

$$\begin{aligned}
\frac{1}{2}\phi(s) &= \sum_{n=1}^\infty (\pi n^2)^{\frac{-s}{2}} \Gamma\left(\frac{s}{2}\right) + \frac{1}{s} + \frac{1}{1-s} \\
&= \sum_{n=1}^\infty \pi^{\frac{-s}{2}} n^{-s} \Gamma\left(\frac{s}{2}\right) + \frac{1}{s} + \frac{1}{1-s} \\
&= \pi^{\frac{-s}{2}} \zeta(s) \Gamma\left(\frac{s}{2}\right) + \frac{1}{s} + \frac{1}{1-s}.
\end{aligned}$$

We already established the integrals making up ϕ were convegent so for all s such that $\mathrm{Re}(s) > 1$, we have that ϕ is entire with respect to s. Thus, we have meromorphic continuuation:

$$\frac{\pi^{\frac{s}{2}}\left(\frac{1}{2}\phi(s) - \dfrac{1}{s} - \dfrac{1}{1-s}\right)}{\Gamma\left(\dfrac{s}{2}\right)} = \zeta(s).$$

Moreover, $\pi^{\frac{s}{2}}, \dfrac{1}{\Gamma\left(\frac{s}{2}\right)}, \phi(s)$ are entire so the poles of ζ are simple and at $s = 0, 1$.

Consider the pole at $s = 0$:

We have $\Gamma\left(\dfrac{s}{2}\right) s$ in the denominator. Replace $\Gamma\left(\dfrac{s}{2}\right) s = 2\left(\dfrac{s}{2}\right)\Gamma\left(\dfrac{s}{2}\right) = 2\Gamma\left(\dfrac{s}{2}+1\right) \neq 0$ as $s \to 0$. So, we have fixed this.

Next, consider the pole at $s = 1$:

To find the residue, we consider

$$\lim_{s\to 1}(s-1)\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left(\frac{1}{2} - \frac{1}{s} - \frac{1}{1-s}\right) = \frac{\pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} = 1.$$

Finally, we need to establish the functional equation:
Let $\Lambda(s) = \dfrac{1}{2}\phi(s) - \dfrac{1}{s} - \dfrac{1}{1-s}$. Since $\dfrac{1}{s}$ and $\dfrac{1}{1-s}$ are invariant replacing $s \to 1-s$, we just need to show that $\phi(s) = \phi(1-s)$.
Recall the functional equation for θ. Then, switch the bounds on ϕ from $t \to \dfrac{1}{t}$. After a little work, we obtain:

$$\begin{aligned}
\phi(s) &= \int_0^1 t^{\frac{-s}{2}} \left(\theta\left(\frac{1}{t}\right) - 1\right)\frac{\mathrm{d}t}{t} + \int_1^\infty t^{\frac{-s}{2}} \left(\theta\left(\frac{1}{t}\right) - \sqrt{t}\right)\frac{\mathrm{d}t}{t} \\
&= \int_0^1 t^{\frac{-s}{2}} \left(\sqrt{t}\theta(t) - 1\right)\frac{\mathrm{d}}{t} + \int_1^\infty t^{\frac{-s}{2}} \left(\sqrt{t}\theta(t) - \sqrt{t}\right)\frac{\mathrm{d}}{t} \\
&= \int_0^1 t^{\frac{1-s}{2}} \left(\theta(t) - \frac{1}{\sqrt{t}}\right)\frac{\mathrm{d}t}{t} + \int_1^\infty t^{\frac{1-s}{2}} \left(\theta(t) - 1\right)\frac{\mathrm{d}t}{t} \\
&= \phi(1-s).
\end{aligned}$$

This completes the proof of the functional equation and, thus, the theorem.

Remark
In a similar way, one can obtain analytic continuuation and functional equations for more general series by inserting Dirichlet characters. Furthermore, the Mellin transform extends to $\mathbb{C}$ easily, i.e. one could define the Mellin transform for the $\mathbb{C}$ case, as well, taking the integral to $i\infty$.