Codes from incidence matrices of graphs

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Incidence matrix of a graph

An incidence matrix for an undirected graph $\Gamma = (V, E)$ is a $|V| \times |E|$ matrix $G = [g_{x,e}]$ with

- rows labelled by the vertices $x \in V$ and
- columns by the edges $e \in E$,

where $g_{x,e} = 1$ if $x \in e$, $g_{x,e} = 0$ if $x \notin e$.

Row span of incidence matrix of a graph

For any prime p let $C_p(G)$ be the row span of G over $\mathbb{F}_p$.

It has been found that for many classes of connected graphs that have some regularity and symmetry, these codes have parameters

$$[|E|, |V| - \varepsilon_p, \delta(\Gamma)]_p$$

where

- $\varepsilon_2 = 1$, $\varepsilon_p = 0, 1$ for p odd;
- $\delta(\Gamma)$ is the minimum degree of Γ;
- the words of minimum weight are precisely the non-zero scalar multiples of the rows of G of weight $\delta(\Gamma)$.

Gap in the weight enumerator

Furthermore, it was found that there is often a gap in the weight enumerator between k and $2(k-1)$, the latter weight arising from the difference of two rows, i.e. there are no words of weight m where

$$k < m < 2(k-1).$$

Comment on the gap in the weight enumerator

This gap occurs for the p-ary code of the desarguesian projective plane $PG_2(\mathbb{F}_q)$, where $q = p^t$; also for other designs from desarguesian geometries $PG_{n,k}(\mathbb{F}_q)$: see [Cho00, LSdV08a, LSdV08b]

But, not always true for non-desarguesian planes: e.g. there are planes of order 16 that have words in this gap: see [GdRK08].

This has also shown that there are affine planes of order 16 whose binary code has words of weight 16 that are not incidence vectors of lines.

Adjacency matrix

Note:

For $\Gamma = (V, E)$, the row span $C_p(\Gamma)$ of a $|V| \times |V|$ adjacency matrix for Γ over $\mathbb{F}_p$ gives linear code of length $|V|$ that may have properties that are of use in classifications or in applications.

However no uniform properties of these codes, other than possibly their dimension over different p, seems to emerge, even for attractive infinite classes of graphs.

Exception: for the line graph $L(\Gamma)$,

$$C_2(L(\Gamma)) \subseteq C_2(G)$$

where G is an incidence matrix for Γ.

Note on earlier work

The code $C_2(G)$ has been referred to in the literature as the bond space or the cut space. See for example, Hakimi and Bredeson [HB68, BH67] for binary codes.
Their interest in the codes was for the application of majority logic decoding.

The codes $C_2(G)^\perp$ were termed graphical codes by Jungnickel and Vanstone and studied for a number of coding properties in [JV96, JV97b, JV99b, JV95, JV99a, JV97a].

Graphs terminology

The **graphs**, $\Gamma = (V, E)$ with vertex set V, $N = |V|$, and edge set E, are undirected with no loops.

- If $x, y \in V$ and x and y are adjacent, $\mathbf{x \sim y}$, and $\mathbf{[x, y]}$ or **xy** is the edge they define.
- A graph is **regular** if all the vertices have the same valency k.
- An **adjacency matrix** $A = [a_{i,j}]$ of Γ is an $N \times N$ matrix with $a_{ij} = 1$ if vertices $v_i \sim v_j$, and $a_{ij} = 0$ otherwise.
- An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{J})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{J}$ is a t-(v, k, λ) **design,** if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks.

Terminology and definitions continued

- The **neighbourhood design $\mathcal{D}(\Gamma)$** of a regular graph Γ is the 1-(N,k,k) symmetric design with points the vertices of Γ and blocks the sets of neighbours of a vertex, for each vertex, i.e. an adjacency matrix of Γ is an incidence matrix for $\mathcal{D}$.
- An **incidence matrix** of Γ is an $N \times |E|$ matrix B with $b_{i,j} = 1$ if the vertex labelled by i is on the edge labelled by j, and $b_{i,j} = 0$ otherwise.
- If Γ is regular with valency k, then $|E| = \frac{Nk}{2}$ and the 1-$(\frac{Nk}{2}, k, 2)$ design with incidence matrix B is called the **incidence design $\mathcal{G}(\Gamma)$** of Γ.
- The **line graph** $L(\Gamma)$ of $\Gamma = (V, E)$ is the graph with vertex set E and e and f in E are adjacent in $L(\Gamma)$ if e and f as edges of Γ share a vertex in V.

Terminology and definitions continued

- The **code $C_F(\mathcal{D})$ of the design** $\mathcal{D}$ over a field F is the space spanned by the incidence vectors of the blocks over F.
- For $X \subseteq \mathcal{P}$, the **incidence vector** in $F^{\mathcal{P}}$ of X is v^X.
- The **code $C_F(\Gamma)$ or $C_p(A)$ of graph Γ** over $\mathbb{F}_p$ is the row span of an adjacency matrix A over $\mathbb{F}_p$. So $C_p(\Gamma) = C_p(\mathcal{D}(\Gamma))$ if Γ is regular.
- If G is an incidence matrix for Γ, $C_p(G)$ denotes the row span of G over F_p. So $C_p(G) = C_p(\mathcal{G}(\Gamma))$ if Γ is regular.
- If G is an incidence matrix for $\Gamma = (V, E)$, L is an adjacency matrix for $L(\Gamma)$, then

$$(G^T)G = L + 2I_{|E|}$$

Some classes of graphs studied

Infinite classes of graphs studied and found, by combinatorial and coding theoretic methods, along with induction, to have the properties described for $C_p(G)$, G an incidence matrix, include:

1. Hamming graphs $H^k(n,m)$ [FKM10, FKM11]

For n, k, m integers, $1 \leq k < n$, the Hamming graph $H^k(n,m)$=(V,E) where

- V is the set of m^n n-tuples of R^n, where R is a set of size m;
- two n-tuples are adjacent if they differ in k coordinate positions.

They are the graphs from the Hamming association scheme.

In particular, the n-cube: $\mathcal{Q}_n = H(n,2) = H^1(n,2)$ $(R = \mathbb{F}_2)$.

Some classes of graphs studied

2. Uniform subset graphs $\Gamma(n,k,m)$

A uniform subset graph $\Gamma(n,k,m) = (V,E)$ where $V = \Omega^{\{k\}}$, where $|\Omega| = n$, and adjacency defined by $a \sim b$ if $|a \cap b| = m$.
The symmetric group $S_n \subseteq \mathrm{Aut}(\Gamma(n,k,m))$.

All classes studied satisfy the properties described, and include:

- the odd graphs $\Gamma(2k+1,k,0)$[FKMa]
- triangular graphs $\Gamma(n,2,1)$ (strongly regular) and $\Gamma(n,2,0)$[FKMc]
- $\Gamma(n,3,m)$ for $m = 0,1,2$.[FKMb]

Some classes of graphs studied, continued

3. Complete multipartite graphs $K_{n_1,n_2,\ldots,n_k}$
   - K_n the complete graph[KMR10]
   - $K_{n,n}$ the complete bipartite graph[KR10]
   - $K_{n,m}$ for $n \neq m$
   - $K_{n_1,n_2,\ldots,n_k}$ where $n_i = n$ for $i = 1, \ldots, k$

Some classes of graphs studied, continued

4. Strongly regular graphs (n, k, λ, μ)

A graph $\Gamma = (V, E)$ is strongly regular with parameters (n, k, λ, μ) if

- $|V| = n$;
- Γ is regular with valency (degree) k;
- for any $P, Q \in V$ such that $P \sim Q$,

$$|\{R \in V \mid R \sim P \,\&\, R \sim Q\}| = \lambda;$$

- for any $P, Q \in V$ such that $P \not\sim Q$,

$$|\{R \in V \mid R \sim P \,\&\, R \sim Q\}| = \mu.$$

Some classes of graphs studied, continued

- **Triangular graphs** $T(n) = L(K_n)$, $n \geq 4$, $(\binom{n}{2}, 2(n-2), n-2, 4)$[KMR10]
- **Paley graphs** $P(q)$, vertex set $\mathbb{F}_q$ where $q \equiv 1 \pmod 4$ and $x \sim y$ if $x - y$ is a non-zero square, $(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$[GK11]
- **Lattice graphs** $L_2(n) = L(K_{n,n})$, the line graph of the complete bipartite graph, $(n^2, 2(n-1), n-2, 2)$[KS08]
- **Symplectic graphs** [KMR],
  $\Gamma_{2m}(q)$ with parameters $(\frac{q^{2m}-1}{q-1}, \frac{q^{2m-1}-1}{q-1} - 1, \frac{q^{2m-2}-1}{q-1} - 2, \frac{q^{2m-2}-1}{q-1})$ and complement
  $\Gamma^c_{2m}(q)$ with parameters $(\frac{q^{2m}-1}{q-1}, q^{2m-1}, q^{2m-2}(q-1), q^{2m-2}(q-1))$ where $m \geq 2$, q a prime power.

Dimension of $C_p(G)$

Result

$\Gamma = (V, E)$ is a connected graph, G an incidence matrix, then

1. $\dim(C_2(G)) = |V| - 1$.
2. *If Γ has a closed path of odd length ≥ 3, then $\dim(C_p(G)) = |V|$ for p odd.*
3. *If Γ is regular, and $\mathcal{G}$ the incidence design, $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\mathcal{G})$.*

Incidence vectors and notation

For $\Gamma = (V, E)$ a graph,

- for $X \subseteq E$, the **incidence vector** in F^E of X is v^X;
- for $u \in V$, $N(u)$ the neighbours of u,

$$\overline{u} = \{uv \mid v \in N(u)\}$$

  where uv or $[u, v]$ denotes an edge;
- for $u \in V$,

$$v^{\overline{u}} = \sum_{e \in \overline{u}} v^e = \sum_{v \in N(u)} v^{uv},$$

  i.e. the row G_u of the incidence matrix G corresponding to u.

Words in $C_p(G)^\perp$

Result

Let Γ be a graph, $L(\Gamma)$ its line graph, and G an incidence matrix for Γ. If $\pi = (x_1, \ldots, x_l)$ is a closed path in Γ, then

1. $w(\pi) = \sum_{i=1}^{l-1} v^{x_i x_{i+1}} + v^{x_l x_1} \in C_2(G)^\perp$;
2. *if $l = 2m$ and*

$$w(\pi) = \sum_{i=1}^{m} v^{x_{2i-1} x_{2i}} - \sum_{i=1}^{m-1} v^{x_{2i} x_{2i+1}} - v^{x_{2m} x_1},$$

   then $w(\pi) \in C_p(G)^\perp$ for all primes p, and if p is odd, $w(\pi) \in C_p(L(\Gamma))$.

Methods of attack for specific classes

The graphs considered all had large automorphism groups, mostly transitive on vertices and on edges.

Method 1: Combinatorial

All the graphs has short paths of even length t, hence producing words of this weight in the dual code $C^{\perp}$.

Form a 1-$(|E|, t, r)$ design of the supports of these words, compute r (the replication number) for this design, and then count incidence with the support of any word of C.

This frequently was good enough to get the minimum weight, and further the minimum words.

Method 2: Induction, linear algebra and coding theory

This works when taking a class for $n \in \mathbb{N}$, by embedding an incidence matrix for $n-1$ in that for n, and using induction.

General method using edge-cuts in graphs

(Joint work with Peter Dankelmann and Bernardo Rodrigues of UKZN)

More general method showing that these properties hold for many classes of well-behaved connected graphs: see [DKR]

If $\Gamma = (V, E)$ is connected and $S \subset E$, let $\Gamma - S = (V, E - S)$.
If $\Gamma - S$ is disconnected then S is called an **edge-cut**.

The **edge-connectivity $\lambda(\Gamma)$** of Γ is the minimum size of an edge-cut.

So $\lambda(\Gamma) \leq \delta(\Gamma)$ (the minimum degree of Γ) since removing all the edges containing a vertex disconnects the graph.

If $\lambda(\Gamma) = \delta(\Gamma)$ and the only edge sets of cardinality $\lambda(\Gamma)$ whose removal disconnects Γ are the sets of edges incident with a vertex of degree $\delta(\Gamma)$, then Γ is called **super-λ**.

Theorem for the binary case:

Theorem

Let $\Gamma = (V, E)$ *be a connected graph,* G *a* $|V| \times |E|$ *incidence matrix for* Γ*. Then*

1. $C_2(G) = [|E|, |V| - 1, \lambda(\Gamma)]_2$;
2. *if* Γ *is* *super-*λ*, then* $C_2(G) = [|E|, |V| - 1, \delta(\Gamma)]_2$*, and the minimum words are the rows of* G *of weight* $\delta(\Gamma)$*.*

Proof

Proof: $C = C_2(G)$ has dimension $|V| - 1$ by Result 1.
Let d be the minimum weight of C.
(1). Let

$$x = \sum_{u \in V} \mu_u v^{\overline{u}} \in C$$

where $\mu_v \in \mathbb{F}_2$, and $\mathrm{wt}(x) = d$. Then

$$x(uv) = \mu_u + \mu_v.$$

So, for every edge $uv \in E$

$$uv \in \mathrm{Supp}(x) \iff \mu_u \neq \mu_v.$$

Proof continued

Let $\Gamma_x = (V, E - \mathrm{Supp}(x))$.
If $u \sim v$ in Γ_x, then $\mu_u + \mu_v = 0$, and so $\mu_u = \mu_v$.
So for any two vertices u and v in the same component of Γ_x we have $\mu_u = \mu_v$.

Thus Γ_x is disconnected since otherwise, if Γ_x were connected, all μ_v would have the same value, μ say, and so $x = \mu \sum_u v^{\overline{u}} = \mu 0$, a contradiction.

Hence $\mathrm{Supp}(x)$ is an edge-cut of Γ, and so $|\mathrm{Supp}(x)| \geq \lambda(\Gamma)$ and $d = \mathrm{wt}(x) \geq \lambda(\Gamma)$.

Proof continued

Now construct a word of weight $\lambda(\Gamma)$.
Let $S \subseteq E$ be a minimal edge-cut of Γ.
Then $\Gamma - S = (V, E - S)$ has V partitioned into two connected components, W and $V - W$ which are such that if $u, v \in W$ and $u \sim v$, then $uv \notin S$, and similarly for $V - W$.
Thus the edges in S are precisely the edges between W and $V - W$, and not those within either of the components.
Let $x = \sum_{u \in V} \mu_u v^{\overline{u}}$, where $\mu_u = 1$ if $u \in W$, and $\mu_u = 0$ if $u \in V - W$.
For an edge $uv \in E$ we have

$$uv \in \mathrm{Supp}(x) \iff \mu_u \neq \mu_v \iff uv \in S.$$

Hence $\mathrm{wt}(x) = |\mathrm{Supp}(x)| = |S| = \lambda(\Gamma)$.
So the minimum weight of C is $\lambda(\Gamma)$.

Proof continued, Γ super-λ

(2). Now suppose Γ is super-λ.
The minimum weight of C is $\lambda(\Gamma) = \delta(\Gamma)$.
Let $x = \sum_{u \in V} \mu_u v^{\overline{u}}$ be a word in C of weight $\delta(\Gamma)$.
Then $\Gamma_x = (V, E - \mathrm{Supp}(x))$ is disconnected, and $\mathrm{Supp}(x)$ is an edge-cut of cardinality $\lambda(\Gamma)$.
Since Γ is super-λ, it follows that Γ_x has exactly two components, one consisting of a single vertex u of degree $\delta(\Gamma)$, and the other component containing the vertices in $V - \{u\}$.
Thus $\mathrm{Supp}(x) = \{uv \mid v \in N(u)\}$ so $x = v^{\overline{u}}$, which proves (2).■

Examples of super-λ

Let $\Gamma = (V, E)$ be a connected k-regular graph.
Then Γ is super-λ if one of the following conditions is satisfied, so $C_2(G)$ has minimum weight k and the words of weight k are the rows of G:

1a Γ is vertex-transitive and has no complete subgraph of order k (Tindell [Tin]);

2a. Γ has diameter at most 2, and in addition Γ has no complete subgraph of order k (Fiol [Fio92]);

3a. Γ is strongly regular with parameters (n, k, λ, μ), and $\mu \geq 1$, $\lambda \leq k - 3$ (follows from 2. above);

4a. Γ is distance-regular and $k > 2$ (Brouwer and Haemers [BH05]);

5a. $k \geq \frac{|V|+1}{2}$ (Kelmans [Kel72]);

6a. Γ has girth g, and $\mathrm{diam}(\Gamma) \leq g - 1$ if g is odd, or $\mathrm{diam}(\Gamma) \leq g - 2$ if g is even. (Fabrega, Fiol [FF89]).

Argument for p odd

The same argument **does not** follow through for p odd (although the result is surely true for most nice classes of graphs).
If $w \in C_p(G)$, p odd, $w \neq 0$, and

$$w = \sum_{x \in V} \mu_x v^{\overline{x}},$$

then $\mathrm{Supp}(w)$ is an edge-cut, but $\Gamma - \mathrm{Supp}(w)$ might not be disconnected.

A modified argument yields a similar but somewhat more restrictive result.

Note: The same argument as in the binary case **does** follow for odd p for Γ connected and bipartite.

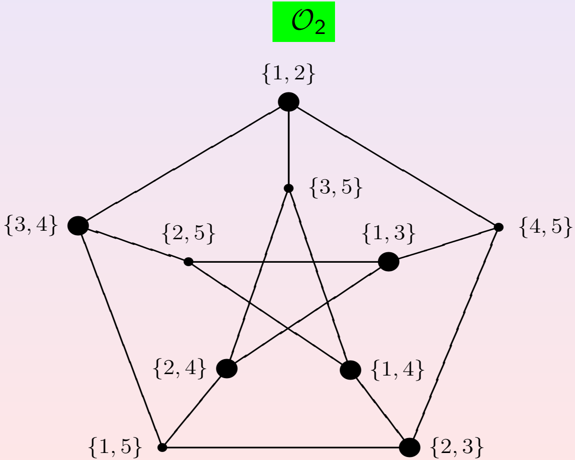
Counter example for p odd: Petersen graph $\mathcal{O}_2$

The Petersen graph, i.e. the smallest odd graph $\mathcal{O}_2 = (V, E)$, where $V = \Omega^{\{2\}}$, and $\Omega = \{1, 2, 3, 4, 5\}$ (strongly regular $(10, 3, 0, 1)$), yields a counterexample: (see [FKMa]).
Here $\overline{x}$ denotes the support of the row of an incidence matrix indexed by $x \in V$. So, for example

$$\overline{\{1,2\}} = \{\{1,2\}\{3,4\}, \{1,2\}\{3,5\}, \{1,2\}\{4,5\}\}.$$

Counter example: Petersen graph $\mathcal{O}_2$

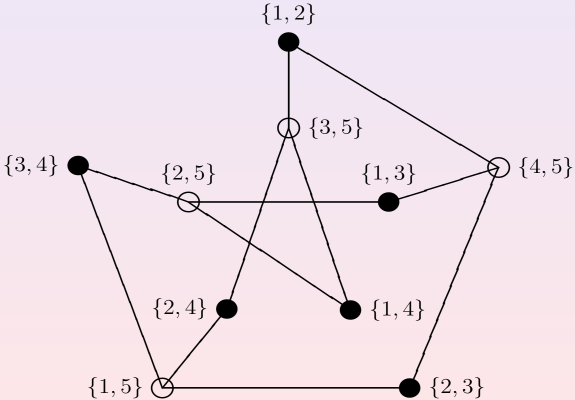
Let $w = v^{\overline{\{1,2\}}} + v^{\overline{\{3,4\}}} + v^{\overline{\{1,3\}}} + v^{\overline{\{2,4\}}} + v^{\overline{\{1,4\}}} + v^{\overline{\{2,3\}}} - \jmath_{15} = v^{\{1,2\}\{3,4\}} + v^{\{1,3\}\{2,4\}} + v^{\{1,4\}\{2,3\}} \in C_p(G)$ for p odd, since $\sum_{x \in V} v^{\overline{x}} = 2\jmath_{15} \in C_p(G)$ and is not 0 for p odd. w is not a row of G.



$\mathcal{O}_2 - \mathrm{Supp}(w)$

So $\mathrm{Supp}(w) = \{\{1,2\}\{3,4\}, \{1,3\}\{2,4\}, \{1,4\}\{2,3\}\}$, $\mathcal{O}_2 - \mathrm{Supp}(w)$ is bipartite (connected) and $\mathrm{Supp}(w)$ is not an edge-cut.

$\mathcal{O}_2 - \mathrm{Supp}(w)$



Bipartite and p odd

For bipartite connected graphs the argument is similar for p odd to that for general connected graphs for $p = 2$:

Theorem

Let $\Gamma = (V, E)$ be a connected bipartite graph, G a $|V| \times |E|$ incidence matrix for Γ, and p any prime. Then

1. $C_p(G) = [|E|, |V| - 1, \lambda(\Gamma)]_p$;
2. *if Γ is super-λ, then $C_p(G) = [|E|, |V| - 1, \delta(\Gamma)]_p$, and the the minimum words are the non-zero scalar multiples of the rows of G of weight $\delta(\Gamma)$.*

General theorem for p odd

For p odd we have:

Theorem

Let $\Gamma = (V, E)$ be a connected k-regular graph that is not bipartite on $|V| = n$ vertices, G an $n \times \frac{nk}{2}$ incidence matrix for Γ, and p an odd prime. If

1. $k \geq (n+3)/2$ *and* $n \geq 6$, *or*
2. Γ *is strongly regular with parameters* (n, k, μ, λ), *where*
    1. $n \geq 7$, $\mu \geq 1$, *and* $1 \leq \lambda \leq k-3$, *or*
    2. $n \geq 11$, $\mu \geq 1$, *and* $\lambda = 0$,

then the code $C_p(G)$ has minimum weight k, and the minimum words are the non-zero scalar multiples of the rows of G.

Restricted edge-connectivity $\lambda'(\Gamma)$

For $\Gamma = (V, E)$ a connected graph, a **restricted edge-cut** is a set $S \subseteq E$ such that

- $\Gamma - S$ is disconnected,
- and no component of $\Gamma - S$ is an isolated vertex.

It was shown in [EH88] that every graph with $|V| \geq 4$ which is not a star has a restricted edge-cut.

The **restricted edge-connectivity** $\lambda'(\Gamma)$ is the minimum number of edges in a restricted edge-cut, if such an edge-cut exists.

If Γ is k-regular with $k \geq 2$ and $|V| \geq 4$, then

$$\lambda'(\Gamma) \leq 2k - 2.$$

(since removing all the edges other than uv through adjacent vertices u and v will produce a restricted edge-cut of size $2(k-1)$).

Gap in the weight enumerator

Theorem

Let $\Gamma = (V, E)$ *be a connected* k*-regular graph with* $|V| \geq 4$*,*
G *an incidence matrix for* Γ*,*
$\lambda(\Gamma) = k$ *and* $\lambda'(\Gamma) > k$*.*

Let W_i *be the number of codewords of weight* i *in* $C_2(G)$*. Then*

- $W_i = 0$ *for* $k + 1 \leq i \leq \lambda'(\Gamma) - 1$*,*
- *and* $W_{\lambda'(\Gamma)} \neq 0$ *if* $\lambda'(\Gamma) > k + 1$*.*

Some classes for which this holds

Corollary

Let $\Gamma = (V, E)$ *be a connected* k*-regular graph and* G *an incidence matrix for* Γ*. If* Γ *satisfies one of the conditions*

1. Γ *is vertex-transitive, and has odd order or does not contain triangles (Xu [Xu00]);*
2. Γ *is edge-transitive and has* $|V| \geq 4$ *(Li and Li [LL99]);*
3. *any two non-adjacent vertices of* Γ *have at least three neighbours in common;*
4. Γ *is strongly regular graph with parameters* (n, k, λ, μ) *with either* $\lambda = 0$ *and* $\mu \geq 2$*, or with* $\lambda \geq 1$ *and* $\mu \geq 3$ *(from 3. above);*

then $C_2(G)$ *has minimum weight* k*, the words of weight* k *are precisely the rows of the incidence matrix, and there are no words of weight* ℓ *such that* $k < \ell < 2k - 2$*.*

Codes from adjacency matrices of line graphs

$\Gamma = (V, E)$, M an $|E| \times |E|$ adjacency matrix for the line graph $L(\Gamma)$.
The rows of M are labelled by the edges $[P, Q] \in E$, which has neighbours:

$$N([P,Q]) = \overline{[P,Q]} = \{[P,R] \mid R \neq Q\} \cup \{[R,Q] \mid R \neq P\}.$$

Recall from Result 2:
If π is a closed path in Γ of even length t, p an odd prime, then $C_p(M)$ has words of weight t.

Binary codes of line graphs

So codes of adjacency matrices of line graphs (of graphs with closed paths of small even length t) over $\mathbb{F}_p$ for p odd have minimum weight at most t, and are not of much interest if t is small, as it is for most interesting classes.

Recall:
if G is an incidence matrix for Γ, M an adjacency matrix for $L(\Gamma)$ then

$$G^T G = M + 2I_e.$$

So

$$C_2(M) \subseteq C_2(G),$$

spanned by the differences of pairs of rows of G.

Binary codes of line graphs

Result

Let $\Gamma = (V, E)$ be a connected graph, G a $|V| \times |E|$ incidence matrix for Γ, and M an adjacency matrix for $L(\Gamma)$. Let $E(G)$ denote the binary code spanned by the differences of all pairs of rows of G. Then

1. $C_2(M) = E(G)$;
2. $C_2(M) = C_2(G)$ *if and only if* $|V|$ *is odd; if* V *is even,* $[C_2(G), C_2(M)] = 1$.

To prove this, make use of the well-known fact that the 2-rank of a symmetric matrix with 0-main-diagonal is always even (see for example [GR01, Proposition 2.1]), and of the fact that $E(G)$ is either $C_2(G)$ or of co-dimension 1 in it.

Binary codes of line graphs

For classes of graphs examined here previously and from results using edge-cuts, it has now been found that the minimum weight of $C_2(M)$ is

- k if $C_2(M) = C_2(G)$;
- $2k - 2$ if not, i.e. $[C_2(G) : C_2(M)] = 1$.

There are no words of weight between k and $2k - 2$ in $C_2(G)$.

Permutation decoding

Permutation decoding, from MacWilliams [Mac64], involves finding a set of automorphisms of the code, called a PD-set.
See MacWilliams and Sloane [MS83, Chapter 16, p. 513] and Huffman [Huf98, Section 8].

Definition

Let C be a t-error-correcting code with information set $\mathcal{I}$ and check set $\mathcal{C}$.

A **PD-set** for C is a set $S \subseteq \mathrm{Aut}(C)$ such that:
every t-set of coordinate positions is moved by at least one member of S into the check positions $\mathcal{C}$.

For $s \leq t$ an s-**PD-set** is a set $S \subseteq \mathrm{Aut}(C)$ such that:
every s-set of coordinate positions is moved by at least one member of S into $\mathcal{C}$.

In [KMM06, Lemma 7] the following was proved:

Result

Let C be a linear code with minimum weight d, $\mathcal{I}$ an information set, $\mathcal{C}$ the corresponding check set and $\mathcal{P} = \mathcal{I} \cup \mathcal{C}$.
Let G be an automorphism group of C, and n the maximum value of $|\mathcal{O} \cap \mathcal{I}|/|\mathcal{O}|$, over the G-orbits $\mathcal{O}$.
If $s = \min(\lceil \frac{1}{n} \rceil - 1, \lfloor \frac{d-1}{2} \rfloor)$, then G is an s-PD-set for C.

This holds for any information set. If the group G is transitive then $|\mathcal{O}|$ is the degree of the group and $|\mathcal{O} \cap \mathcal{I}|$ is the dimension of the code. This is applicable to codes from incidence matrices of connected regular graphs with automorphism groups transitive on edges:

Result ([FKMb])

Let $\Gamma = (V, E)$ be a regular k-graph with $A = \mathrm{Aut}(\Gamma)$ transitive on edges, and M be an incidence matrix for Γ.

If $C = C_p(M) = [|E|, |V| - \varepsilon, k]_p$, where $\varepsilon \in \{0, 1, \ldots, |V| - 1\}$, then any transitive subgroup of A will serve as a PD-set for full error correction for C.

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