Rigidity Theorems for a Class of Affine Resolvable Designs

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Abstract

The affine resolvable 2-$(27, 9, 4)$ designs were classified by Lam and Tonchev [9, 10]. We use their construction of the designs to examine the ternary codes of the designs and show, using Magma [3], that each of the codes, apart from two, contains, amongst its constant weight-9 codewords, a copy of the ternary code of the affine geometry design of points and planes in $AG_3(F_3)$. We also show how the ternary codes of the 68 designs and of their dual designs, together with properties of the automorphism groups of the designs, can be used to characterize the designs.

1 Introduction

There are many designs with the same parameters as those of the designs of points and r-dimensional subspaces of a finite projective geometry $PG_m(F_q)$, or of points and r-flats of a finite affine geometry $AG_m(F_q)$, where r is fixed and $1 \leq r \leq m$. The designs from the geometries have the largest automorphism groups (see [7]) and the codes associated with these designs, i.e. the span of the incidence vectors of the blocks over the prime field of characteristic dividing q, are members of the class of generalized Reed-Muller codes: see [1, Chapter 5]. It is a long standing conjecture of

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Hamada [5] that the code of the design from the geometry has the smallest dimension amongst those of the same parameters. This conjecture has not been contradicted, and in some cases has been proved to be true: see [1] or [8] for more on this. The proof of this conjecture would provide a partial "rigidity theorem" for such designs since it would characterize the designs from geometries as being those whose codes have the smallest dimension for their class. A further rigidity theorem would hold if they were the unique designs with codes of this minimal dimension, but this is not true in general: see [12].

The minimum weight of the codes from the projective or affine geometry design is always the block size and the words of minimum weight are simply multiples of the incidence vectors of the blocks, as was shown by Delsarte and Goethals and others in a series of papers: see [1, Chapter 5] for full references to this work. For other designs the codes do not necessarily have this property: for some parameters (see [6] or [1, Corollary 7.5.1]) we know that this property characterises the design from the geometry, leading to a rigidity theorem for this kind of design: see [1, Theorem 8.2.1] for an example of this.

A question that can be raised is the following: given a design with parameters that of a projective or affine geometry design, does the code of the design over the "correct" prime field contain a copy of the geometry design amongst the supports of its constant vectors of weight the block size of the design? There are certainly cases where this is known not to be the case: for example, projective planes are always characterized by their codes, and also certain 2-$(31, 7, 7)$ designs (see Tonchev [12] or Delsarte and Goethals [4]) and some other designs (see [2, Section 5]). However, the question has not been answered for all classes with the parameters of a design from a finite geometry; the result [1, Theorem 8.2.1] quoted above is one case where it is known to hold.

We look here at the affine resolvable 2-$(27, 9, 4)$ designs, as classified recently by Lam and Tonchev [9, 10]. There are 68 of these, including the design $AG_{3,2}(F_3)$ of points and planes in $AG_3(F_3)$. Lam and Tonchev determined the automorphism groups and the dimensions of the ternary codes of the designs and noted that Hamada's conjecture was not contradicted and that $AG_{3,2}(F_3)$ is in fact the only design whose ternary code has the minimal dimension 10.

Using Magma [3] we constructed the code of each design, and collected all the constant weight-9 codewords. Then, using techniques described in Section 3, we were able to find, for all but two of the designs, a set of 39 of these codewords whose supports formed the affine geometry design. The two designs that did not share this property, designs ##15 and 18 in the numbering of [9, 10], did not have the property for the simple reason that they each had exactly 39 constant weight-9 words, and these were then just the incidence vectors of the design itself. Thus no other design of these parameters resides amongst the constant codewords in these two cases. Since the ternary codes of these designs are all self orthogonal, this also means that all but two of the designs can be found amongst the constant weight-9 codewords of the dual code of the ternary code of the design $AG_{3,2}(F_3)$. We were also able to determine which designs can be found amongst the constant codewords of weight 9 of the other designs: this can be found in Section 3.

We have recorded all the blocks of each design, along with the blocks of the included affine geometry design, at the www site:

`http://www.math.clemson.edu/faculty/Key`

under the heading "affine resolvable 2-$(27, 9, 4)$ designs". Alternatively, printed versions can be obtained from either author or in a Clemson Technical Report.

Finally we also looked at the ternary codes of the dual designs, in the hopes that these would be different for the distinct designs. We were able to obtain all the weight enumerators and found that all the designs can be distinguished from properties of either the group, the ternary code, or the ternary code of the dual design: see Proposition 3. We show how this can be done in the final section. Since an affine resolvable design can always be extended to a symmetric 2-$(40, 13, 4)$ design in the standard way by adjoining a projective plane of order 3 at infinity, the code of each of these dual designs (which are just 1-designs) are subcodes of the shortened code of the dual of the symmetric design. Hence again only the ternary code gives any hope of classification. A recent paper by Mavron and Tonchev [11] studies some other distinguishing properties of the 68 designs. Note that Tonchev and Weishaar [13] characterized all the 80 distinct Steiner triple systems on 15 points by the binary codes of their dual designs.

2 Terminology and notation

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a t-(v, k, λ) design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. The **dual** structure of $\mathcal{D}$ is $\mathcal{D}^t = (\mathcal{B}, \mathcal{P}, \mathcal{I})$. Thus the transpose of an incidence matrix for $\mathcal{D}$ is an incidence matrix for $\mathcal{D}^t$.

We write $AG_{m,n}(F_q)$ for the the 2-(v, k, λ) design of points and n-flats (cosets of dimension n) in the affine geometry $AG_m(F_q)$, where

$$v = q^m,\ k = q^n,\ \lambda = \frac{(q^{m-1} - 1) \ldots (q^{m+1-n} - 1)}{(q^{n-1} - 1) \ldots (q - 1)}.$$

The code C_F of the design $\mathcal{D}$ over the finite field F is the space spanned by the incidence vectors of the blocks over F. We take F to be a prime field F_p, in which case we write also C_p for C_F, and refer to the dimension of C_p as the **p-rank** of $\mathcal{D}$; in the case of the designs from finite geometries, p will be the same as the characteristic of the field over which the geometry is defined. In the general case of a 2-design, the prime must divide the order of the design, i.e. $r - \lambda$, where r is the replication number for the design, that is, the number of blocks through a point. If the point set of $\mathcal{D}$ is denoted by $\mathcal{P}$ and the block set by $\mathcal{B}$, and if $\mathcal{Q}$ is any subset of $\mathcal{P}$, then we will denote the incidence vector of $\mathcal{Q}$ by $v^{\mathcal{Q}}$. Thus $C_F = \langle v^B \mid B \in \mathcal{B} \rangle$, and is a subspace of $F^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to F. For any code C, the **dual** or **orthogonal** code $C^\perp$ is the orthogonal under the standard inner product. If a linear code over a field of order q is of length n, dimension k, and minimum weight d, then we write $[n, k, d]_q$ to show this information. If c is a codeword then the **support** of c is the set of non-zero coordinate positions of c. A **constant word** in the code is a codeword, all of whose coordinate entries are either 0 or 1. The all-one vector will be denoted by $\boldsymbol{\jmath}$, and is the constant vector of weight the length of the code. Two linear codes of the same length and over the same field are **equivalent** if each can be obtained from the other by permuting the coordinate positions and multiplying each coordinate position by a non-zero field element. They are **isomorphic** if they can be obtained from one another by permuting the coordinate positions.

Suppose that $\mathcal{D}$ is a 2-(v, k, λ) design with b blocks and with replication number r. Let $\mathcal{S}$ be any set of points of $\mathcal{D}$. A block is called an **i-secant** if the block meets $\mathcal{S}$ in i points. Let $|\mathcal{S}| = s$. For $i = 0, \ldots, k$, let x_i denote the number of i-secants to $\mathcal{S}$; for a fixed point $z \in \mathcal{S}$, let z_i (or $z_i(z)$) be the number of i-secants passing through z. Standard counting gives the following equations:

$$\sum_{i=0}^{k} x_i = b; \quad \sum_{i=1}^{k} i x_i = sr; \quad \sum_{i=2}^{k} i(i-1) x_i = s(s-1)\lambda, \tag{1}$$

and

$$\sum_{i=1}^{k} z_i = r; \quad \sum_{i=2}^{k} (i-1) z_i = (s-1)\lambda. \tag{2}$$

When $\mathcal{S}$ is the support of a word in the dual code of the design, clearly $x_1 = 0$, so from the last two equations of (1) we obtain

$$\sum_{i=3}^{k} i(i-2) x_i = s((s-1)\lambda - r), \tag{3}$$

and, with $z_1 = 0$, from (2) we obtain

$$\sum_{i=3}^{k}(i-2)z_i = (s-1)\lambda - r. \tag{4}$$

A **parallel class** of blocks of a 2-(v, k, λ) design is a set of pairwise disjoint blocks that partitions the point set. For this to exist, we must have k a divisor of v. Further, the design is **resolvable** if the block set can be partitioned into r disjoint parallel classes, in which case any such partition is called a **resolution**. A resolvable design is **affine resolvable** if any two blocks that are not in the same parallel class intersect in exactly $\frac{k^2}{v}$ points. Thus an affine resolvable design has a unique resolution. Clearly the designs from affine geometries are affine resolvable.

Lam and Tonchev [9, 10] proved the following:

Result 1 (Lam & Tonchev) *There are exactly* 68 *non-isomorphic affine resolvable* 2-$(27, 9, 4)$ *designs; these are given in* [*9*, *10*].

3 Results

For each of the designs given in [9, 10] we constructed the design and the ternary code of length 27 spanned by the incidence vectors of the blocks of the design. Using Magma we collected the constant words of weight 9, and formed the set $\mathcal{S}$ of supports of these vectors. Members of $\mathcal{S}$ will now be called blocks. The size of $\mathcal{S}$ we found for each design are given below, where we use the numbering given in [9, 10].

We then collected all the members, $\mathcal{T}$, of $\mathcal{S}$ through a point p_1; then we found the number of blocks in $\mathcal{T}$ through each of the other 26 points. Choosing a point p_2 with the smallest such set, we looked for sets of four blocks that met each other mutually in three points, and such that their intersection had size three; this is the defining property of the affine geometry design. Using a choice of these four blocks as a starting point, we then chose a set of four blocks with this property for another of the points distinct from p_1, say p_3. With these seven blocks covering the blocks through two triples of points including p_1, we then went on to a further point p_4 and found the quadruples of blocks with the given restriction through this point. Next we formed the ternary code spanned by the incidence vectors of the first seven blocks and this new choice. This code should be a $[27, 7, 9]_3$ code with exactly 13 constant vectors of weight 9 whose supports form the blocks through the point p_1; if the code did not have these properties, a new quadruple of the blocks through p_1 and p_4 was chosen and tested. If none of these gave the correct code, it was necessary to go back to the choice of blocks through p_3 and repeat the process. In practice we always finally found the 13 blocks giving the $[27,7,9]_3$ code with the above property, for each of the designs.

Having found the 13 blocks through p_1, we took a new point, say p_2, and looked at correctly intersecting quadruples of blocks through p_2. A suitable choice, when adjoined to the 13 blocks through p_1 as already found, was achieved when the code spanned by the incidence vectors of the 13 blocks and of the new set was a $[27,9,9]_3$; this was always possible. This then gave 21 blocks, and the set was completed by simply looking through the remaining blocks of $\mathcal{S}$ and forming the code spanned by including the incidence vector of this block, until the correct $[27,10,9]_3$ was found with the words of weight 9 forming the design. This worked in all cases where the size of $\mathcal{S}$ was greater than 39.

Thus we proved

Proposition 1 *All but two of the affine resolvable* 2-$(27,9,4)$ *designs contain a copy of the affine geometry design* $AG_{3,2}(F_3)$ *amongst the constant weight-*9 *codewords of their ternary code of length* 27*. The two that do not have this property (## 15 and 18) have only* 39 *constant weight-*9 *codewords.*

We can also make a few observations about codewords of low weight for an affine resolvable 2-$(27,9,4)$ design:

Proposition 2 *Let C be the ternary code of an affine resolvable* 2-$(27,9,4)$ *design $\mathcal{D}$. Then $C \subseteq C^{\perp}$ and*

(i) the minimum weight of $C^{\perp}$ is at least 6*;*

(ii) a non-zero constant word of $C^{\perp}$ has weight 9, 18 *or* 27*;*

(iii) if the minimum weight of C is 6 *then C contains a constant word of weight* 9 *whose support is not a block;*

(iv) unless $\mathcal{D}$ is the affine geometry design, C has words of weight 9 *whose supports are not blocks.*

Proof: Since blocks meet in 0 or 3 points, it is clear that $C \subseteq C^{\perp}$. Also note that the all-one vector $\jmath$ is in C, since the sum of all the blocks in a parallel class will give $\jmath$. From this it follows that any vector in C will have weight divisible by 3, and the sum of the non-zero coordinate entries of any word of $C^{\perp}$ must be 0.

To prove (i), let c be a non-zero constant vector in $C^{\perp}$ of weight $s > 0$ and support $\mathcal{S}$. Using the notation of Equations (1) to (4), we have, from Equation (4), $s \geq \frac{r}{\lambda} + 1$, i.e. $s \geq 5$. So assume $s = 5$. Now from Equations (2) and (4), we have

$$z_2 = r - \sum_{i=3} z_i \geq r - \sum_{i=3} (i-2) z_i = r - \{(s-1)\lambda - r\} = 2r - (s-1)\lambda,$$

which becomes $z_2 \geq 26 - 16 = 10$, for any point of $\mathcal{S}$. Looking at the entries in the codeword $c \in C^\perp$, then if $\mathcal{S} = \{p_i : 1 \leq i \leq 5\}$, since $\jmath \in C^\perp$, we must, essentially, have entries $+1$ at four of the points, say p_i for $i = 1$ to 4, and -1 at the other, p_5. Every 2-secant through p_1 must pass through p_5; there are only four of these, and hence not all of the ten can pass through p_5, and we have a contradiction. Thus we have proved (i).

To prove (ii), we again use Equations (1) to (4), with the notation of that paragraph. Suppose c is a constant vector in $C^\perp$ of weight s. Since $(c, \jmath) = 0$, s is divisible by 3, and since $\jmath \in C^\perp$, we can assume that $s \leq 12$. Only x_0, x_3, x_6 and x_9 can be non-zero. We have from (i) that $s \geq 6$. Equations (1) with $s = 6$ or 12 have no positive interger solutions. If $s = 9$ and c is not the support of a block (i.e. $x_9 = 0$), then the equations show that $x_6 = 3$, $x_3 = 33$ and $x_0 = 3$.

To prove (iii), suppose C has minimum weight 6, and let c be a codeword of weight 6. Then c is not constant, by (ii), and hence has three coordinate positions, say $\{p_1, p_2, p_3\}$ with entry 1, and three, $\{p_4, p_5, p_6\}$ with entry -1. Let B_i for $i = 1, 2, 3, 4$ be the four blocks of the design through p_1 and p_2.

If one of the blocks, say B_1, also contains p_3 but none of the others, then $v^{B_1} - c$ will be a constant vector of weight 9, but not a block as it has six points in common with B_1. If B_1 contains the full support of c then since B_2 can only meet B_1 in three points, the other point must be p_3 and we can argue as for B_1. Otherwise, none of the blocks B_i contains the set $\{p_1, p_2, p_3\}$, and so each block must contain two of the points from $\{p_4, p_5, p_6\}$. But there are only three distinct choices of pairs from this set, and so two of the blocks would have to intersect in four points, a contradiction.

For (iv), suppose $\mathcal{D} \neq AG_{3,2}(F_3)$. Then there must be a pair of points $\{x, y\}$ such that the four blocks B_i, for $i = 1, 2, 3, 4$ through them do not intersect mutually in three points. Let $B_1 \cap B_2 = \{x, y, z\}$. There are essentially two possibilities: either there is a third block, say B_3, that contains z, or any three of the blocks intersect only in $\{x, y\}$.

In the first event $\jmath - \sum_{i=1}^4 v^{B_i}$ is a vector of weight 6, and hence by (iii) there is a word of weight 9 whose support is not a block; in the second event, $\jmath - \sum_{i=1}^4 v^{B_i}$ is a non-constant vector of weight 9, and it follows that its support is not a block, since otherwise we would contradict (i). □

We give below, in Figure 1, a summary of the Magma output that gave the number of constant weight-9 words, and the weight enumerator of the design's ternary code. The numbering is that of [9, 10], as previously, and note that the ## 31, 41 and 67 refer to the new designs of these numbers, as given in the correction of [10]. Notice that the codes for the designs ## 15 and 18 are the only ones with minimum weight 9 apart from the affine geometry design, # 1. The Magma notation $< x, y >$ indicates that there are y words of weight x.

Note: 1. In every case the design formed by taking all the constant weight-9 vectors from the code or from its dual was at least a 1-design.

2. The codes with the same weight distribution were not necessarily equivalent, although in most cases they were. In fact only in (c) and (f) above did we find, using Magma, inequivalent codes, and the equivalent sets are shown in these two cases. For each equivalence class, the 1-designs formed by the supports of the constant weight-9 codewords were isomorphic, and thus the codes are actually isomorphic. Thus we have the incidence vectors of the blocks of a copy of each of the designs in the equivalence class amongst the constant weight-9 words of the code of any of the other designs in the class. Thus, for example, the code of design #9 has copies of all the other designs in class (d) to be found amongst its constant weight-9 vectors, along with the affine geometry design. In fact, using the blocks of the affine geometry design that we found inside the code of any of these designs, we were able to build from this to establish the containment relations amongst the codes of the designs. Essentially every containment that appears to be possible from the weight distributions, does occurr. Thus, for example, the code of design #59 (in class (g)) has amongst the supports of its 165 constant weight-9 vectors all the other designs in class (g) and all the designs in classes (d), (c), (c*), (b) and (a); however design #14 (in class (e)) includes all the designs from (e), (d), (b) and (a), but not those from (c) nor (c*). This provides a use for the partial "rigidity" property of the affine geometry design in that its code's containment inside the codes of the other designs is used to establish inclusions amongst their codes. We show the inclusions in Figure 2. In all there are nine inequivalent codes to be found from the ternary codes of the designs, but only seven weight distributions and seven complete weight enumerators.

3. The dual code of the ternary code of $AG_{3,2}(F_3)$ has 975 constant weight-9 words, and our computations have thus shown that all but two (## 15 and 18) of the 68 designs can be found amongst these words.

4. The weight enumerator of both of the dual codes of the ternary codes of designs ## 15 and 18 is:

```
[<0,1>, <8,702>, <9,1482>, <11,33696>, <12,44928>, <14, 411480>,
<15,356616>, <17,1388556>, <18,771420>, <20,1166724>, <21,388908>,
<23,185328>, <24,30888>, <26,2160>, <27,80>]
```

```
(a) Design #{1} 3-rank design= 10
weight distribution code=
[ <0, 1>, <9, 78>, <12, 1404>, <15, 14040>,
<18, 27300>, <21, 15444>, <24, 702>,<27, 80> ]
no of constant weight-9's 39
no of constant weight-9's of dual code 975
--------------------------------------------------
(b) Designs ##{2,4} 3-rank design= 11
weight distribution code=
[<0, 1>, <6, 18>, <9, 114>, <12, 4806>, <15, 40572>,
<18, 84090>, <21, 44604>, <24, 2826>, <27, 116> ]
no of constant weight-9's 57
no of constant weight-9's of dual code 489
--------------------------------------------------
(c) Designs ##{3,5,10-13} and (c*)##{6-8,16}
3-rank design= 12
weight distribution code=
[ <0, 1>, <6, 72>, <9, 222>, <12, 15012>, <15,120168>,
<18, 254460>, <21, 132084>, <24, 9198>, <27, 224> ]
no of constant weight-9's 111
no of constant weight-9's of dual code 327
--------------------------------------------------
(d) Designs ## {9,19,20,23,29} 3-rank design= 12
weight distribution code=
[ <0, 1>, <6, 36>, <9, 366>, <12, 14904>, <15,119808>,
<18, 255360>, <21, 131220>, <24, 9594>, <27, 152> ]
no of constant weight-9's 75
no of constant weight-9's of dual code 219
--------------------------------------------------
(e) Designs ## {14,25,57,60-62,64,65} 3-rank design= 13
weight distribution code=
[ <0, 1>, <6, 54>, <9, 1266>, <12, 45090>, <15,357156>,
 <18, 770070>, <21, 390204>, <24, 30294>, <27, 188> ]
no of constant weight-9's of code and of dual 93
--------------------------------------------------
(f) Designs #{15} and (f*) #{18} 3-rank design= 13
weight distribution code=
 [ <0, 1>, <9, 1482>, <12, 44928>, <15, 356616>,
 <18,771420>, <21, 388908>, <24, 30888>, <27, 80> ]
no of constant weight-9's of code and of dual 39
--------------------------------------------------
(g) Designs ##{17,21,22,24,26-28,30-56,58,59,63,66-68}
3-rank design= 13
weight distribution code=
 [ <0, 1>, <6, 126>, <9, 978>, <12, 45306>, <15,357876>,
 <18, 768270>, <21, 391932>, <24, 29502>, <27, 332> ]
no of constant weight-9's of code and of dual 165
```

Figure 1: Weight distributions of the ternary codes

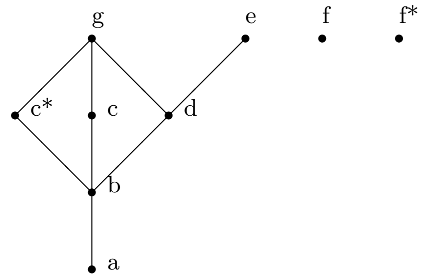

Figure 2: Code inclusions

We found that the supports of the words of weight 8, 9, 11 and 12 formed 2-(27,8,28), 2-(27,9,76), 2-(27,11,2640) and 2-(27, 12, 4224) designs respectively, for each of these dual codes, and that the designs from the different codes were not isomorphic. We did not check all the other weights as the number of blocks became too large, but we did note that for weight 24 the structure was not a 2-design. Note that the codes do not satisfy the conditions of the Assmus-Mattson theorem (see, for example, [1, Theorem 2.11.2]). In the case of the 2-(27,8,28) designs, the size of $\lambda = 28$ is the smallest for a 2-design with this number of points, $v = 27$, and this block size, $k = 8$.

4 The ternary codes of the dual designs

In the hopes of finding another invariant, we constructed the 68 dual designs and their ternary codes. The designs can now all be distinguished, either by the size of the automorphism group, or by the lengths of the orbits on points (as listed in Table 1 of [9, 10]), or by properties of the code, or by the code of the dual design. The latter properties are either the dimension, or the weight distribution, or the nature of the designs formed by taking the supports of words of a fixed weight in the code of the dual design, as will be explained below.

First of all, we attempt to use the weight distributions of the code of the dual designs. We list below, in Table 1, tables giving some characteristics of the ternary code of the dual design of each of the 68 designs that will be the first step in distinguishing the designs. In fact we observed that only the numbers of weight-15 and weight-37 vectors need be given to distinguish the weight distribution, apart from #1 and #6, which are easily distinguished anyway. The full weight distributions can be found at the web site with address as given in the Section 1. The tables below can be used in conjunction with Table 1 of [9, 10] to distinguish, in most cases, the 2-(27,9,4) design. Thus for example designs #32 and #33 are distinguished by this table but not by the order or structure of their automorphism groups, nor by the ternary codes of the designs. On the other hand, #45, #48 and #49, for example, are indistinguishable by any of the given invariants so far: we will show how they can be distinguished from further properties of these codes below. In the tables, superscripts denote designs with the same weight distribution. Note again that the numbers 31, 41 and 67 refer to the new designs of these numbers, as given in the correction of [10]. In Table 1 the designs are listed horizontally with the number of weight-15 and weight-37 vectors listed in the corresponding column.

$\mathcal{D}$	1	2	3^a	4	5	6	7	8^a	9	10	11	12
15	0	0	48	12	216	0	36	48	90	108	288	168
37	0	54	48	12	216	0	90	48	216	90	216	132

$\mathcal{D}$	13	14	15	16	17	18	19	20	21^b	22^c	23	24
15	156	486	624	24	306	624	108	120	396	450	120	372
37	174	486	312	24	576	104	102	174	450	576	72	228

$\mathcal{D}$	25	26^b	27	28	29	30	31	32	33	34^d	35	36^d
15	516	396	468	540	120	468	492	492	468	456	480	456
37	276	450	162	360	40	288	312	216	192	294	318	294

$\mathcal{D}$	37	38^c	39	40	41	42^e	43	44	45^f	46^e	47^e	48^f
15	480	450	408	432	496	556	408	468	544	556	556	544
37	354	576	246	174	218	260	150	192	302	260	260	302

$\mathcal{D}$	49^f	50	51^g	52^g	53	54^g	55^g	56	57	58	59^h	60
15	544	480	676	676	604	676	676	492	494	420	444	532
37	302	258	344	344	344	344	344	186	444	92	116	220

$\mathcal{D}$	61	62	63^i	64	65	66^h	67	68^i
15	532	528	600	572	532	444	348	600
37	342	146	174	152	310	116	108	174

Table 1: Weights in codes of dual designs

The designs ## $\{34, 36\}$, $\{42, 46, 47\}$, $\{45, 48, 49\}$ and $\{51, 52, 54, 55\}$ are apparently indistinguishable within their groupings from the data that we currently have, since their codes have the same weight enumerators and the same complete weight enumerators within the group. To distinguish these we found the cardinality of the set of supports of fixed-weight vectors

design	weight	support	weight	support
34	19	10671		
36	19	10669		
42	37	35		
46	37	29		
47	37	33		
45	19	10565		
48	19	10562		
49	19	10564		
51	19	11077		
52	19	11073		
54	19	11075	37	40
55	19	11075	37	39

Table 2: Weights and supports

(for various chosen weights) from the code of the dual design. It turned out that weights could be found that would give different cardinalities for these support sets for the different codes in the grouping. Thus for each of the four groupings we could find weights that would distinguish the individual designs. In Table 2, the first column gives the design number, the second the weight chosen, and the third the cardinality of the set of supports of vectors of the chosen weight, with the last two columns repeated in case a single weight does not distinguish all the designs in the set. We record the weight distributions of these codes in the appendix.

Thus the codes of the 68 dual designs are all inequivalent, although we were not able to check this directly with Magma as the computations took too long. These inequivalences can be contrasted with the many equivalences found amongst the codes of the 68 affine 2-designs.

We summarize what we have found in Proposition 3 below.

Proposition 3 *The* 68 *affine resolvable* 2-$(27, 9, 4)$ *designs can be distinguished either by the order of their automorphism group, or by the point orbits of the group, or by the ternary code of the design or by the ternary code of the dual design.*

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5 Appendix

We give here the full weight distributions of the ternary codes of the dual designs for the designs for which neither the weight distributions, nor the automorphism groups, are sufficient to distinguish them. As always, the numbering refers to that in [9, 10].

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{34,36}
[ <0, 1>, <6, 4>, <9, 12>, <12, 144>, <13, 78>, <15, 456>, <16, 1320>,
<18, 9300>, <19, 21354>, <21, 49608>, <22, 184848>, <24, 154866>,
<25, 396042>, <27, 216746>, <28, 341232>, <30, 93294>, <31, 109428>,
<33, 6836>, <34, 8286>, <36, 174>, <37, 294> ]
{42,46,47}
[ <0, 1>, <6, 6>, <9, 4>, <12, 142>, <13, 98>, <15, 556>, <16, 1232>, <18,
9556>, <19, 21428>, <21, 47560>, <22, 184646>, <24, 158980>, <25, 397136>,
<27, 212694>, <28, 339356>, <30, 95560>, <31, 110690>, <33, 6076>,
<34, 8036>, <36, 306>, <37, 260> ]
{45,48,49}
[ <0, 1>, <6, 4>, <9, 12>, <12, 124>, <13, 86>, <15, 544>, <16, 1328>,
<18, 9280>, <19, 21146>, <21, 49000>, <22, 185480>, <24, 156526>,
<25, 395162>, <27, 214626>, <28, 341864>, <30, 94786>, <31, 109220>,
<33, 6276>, <34, 8294>, <36, 262>, <37, 302> ]
{51,52,54,55}
[ <0, 1>, <6, 2>, <9, 8>, <12, 106>, <13, 74>, <15, 676>, <16, 1208>,
<18, 8896>, <19, 22160>, <21, 49576>, <22, 183074>, <24,156484>,
<25, 397508>, <27, 213678>, <28, 341132>, <30, 95848>, <31, 109046>,
<33, 5864>, <34, 8336>, <36, 302>, <37, 344> ]
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