Binary codes from reflexive uniform subset graphs on 3-sets *

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Abstract

We examine the binary codes $C_2(A_i + I)$ from matrices $A_i + I$ where A_i is an adjacency matrix of a uniform subset graph $\Gamma(n, 3, i)$ of 3-subsets of a set of size n with adjacency defined by subsets meeting in i elements of Ω, where $0 \leq i \leq 2$. Most of the main parameters are obtained; the hulls, the duals, and other subcodes of the $C_2(A_i + I)$ are also examined. We obtain partial PD-sets for some of the codes, for permutation decoding.

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1 Introduction

Codes from the row span over finite fields of incidence matrices of regular graphs have been shown to have uniform properties that can result in the graphs being retrieved from the code: see [**?**, **?**] for the general results concerning these codes and for references to previous work on various classes of graphs that led to formulation of the general result. In contrast, codes from adjacency matrices for graphs have been found to have no uniform properties in general, and the various classes appear to need to be examined separately, although similar techniques can be used over various classes. In particular, we have observed that for uniform subset graphs $\Gamma(n, k, r) = (V, E)$ where the vertices V are k-subsets of a set of size n, with adjacency defined by the k-subsets meeting in r points, the codes from the adjacency matrix over any field are intimately related to a set of k codes on V, denoted by W_i for $1 \leq i \leq k - 1$, and $W_\Pi \subseteq W_i^\perp$, that are defined independently of the actual graph, and that can be studied separately.

In this paper we examine binary codes from adjacency matrices from the uniform subset graphs on 3-sets: let $\Gamma_n^i = \Gamma(n, 3, i) = (V, E^i)$ denote the uniform subset graph with V the set of 3-subsets of $\Omega = \{1, \ldots, n\}$, for $i = 0, 1, 2$, where adjacency in Γ_n^i is defined by vertices being adjacent if the 3-subsets meet in i points. The binary codes $C_2(A_i)$ from the row span over $\mathbb{F}_2$ of adjacency matrices A_i for these graphs were examined in [**?**], and the ternary codes in [**?**]. We look here at the binary codes $C_2(A_i + I)$ from the matrices $A_i + I$, these being the adjacency matrices of the reflexive graph $\mathcal{R}\Gamma_n^i$, which is obtained from Γ_n^i by including a loop at every vertex. The binary codes from $A_i + I$ have similarities with those from A_i (for example, $C_2(A_i)^\perp \subseteq C_2(A_i + I)$) and results from [**?**] can be used to establish some results here. However, there are major differences that make these codes worthy of study, in particular that all of the codes have minimum weight at least $n-2$, whereas some of the $C_2(A_i)$ are the full space $\mathbb{F}_2^{|V|}$. This also applies to some other graphs: see [**?**]. In a separate paper [**?**] we have similar results for the ternary codes, following the work of [**?**].

We summarize our main results concerning the codes $C_2(A_i + I)$ from the row span over $\mathbb{F}_2$ of $A_i + I$ below. The row of $A_i + I$ corresponding to the vertex x is denoted by s_x^i, and the row of A_i corresponding to the

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vertex x by r_x^i, so $s_x^i = r_x^i + v^x$. Other notation used can be found in the following sections, including the codes W_1, W_2, W_Π mentioned above. A summary of these results can be found in Table **??** at the end of the paper.

Theorem 1. *For $n \geq 7$ let A_i be an adjacency matrix for the uniform subset graph Γ_n^i on 3-sets, $C_i = C_2(A_i+I)$, for $i = 0, 1, 2$.*

1. *For n odd, C_0 is a $[\binom{n}{3}, \binom{n}{2}, n-2]_2$ code; for $n \geq 9$ the minimum words are the $w_{a,b} = \sum_{c \in \Omega \setminus \{a,b\}} v^{\{a,b,c\}}$ for $a, b \in \Omega$.*

   For $n \equiv 0 \pmod 4$ C_0 is a $[\binom{n}{3}, \binom{n-1}{2} + 1, d]_2$ code with $n - 1 \leq d \leq 3n - 8$.

   For $n \equiv 2 \pmod 4$, C_0 is a self-orthogonal $[\binom{n}{3}, \binom{n-1}{2}, d]_2$ code with $n \leq d \leq 4n - 16$. C_0 is doubly-even if $n \equiv 2 \pmod 8$.

   The minimum weight of $C_0^\perp$ is 8.

2. *C_1 is a $[\binom{n}{3}, n, \binom{n-1}{2}]_2$ code, and $C_1 = \langle w_a \mid a \in \Omega \rangle$, where $w_a = \sum_{b,c \in \Omega \setminus \{a\}} v^{\{a,b,c\}}$. It has weight distribution given by $\binom{n}{r}$ words of weight $n_r = r\binom{n-r}{2} + \binom{r}{3}$ for each $0 \leq r \leq n$. For $n \geq 8$, the minimum words are the w_a. For $n = 7$ there are a further 21 words from $r = 5$.*

   The set $\mathcal{I} = \{\{i, n-1, n\} \mid 1 \leq i \leq n-2\} \cup \{\{n-3, n-2, n-1\}, \{n-3, n-2, n\}\}$ is an information set for C_1 for all $n \geq 7$. For $n \geq 8$, and for this information set, the sequence of automorphisms $\{Id, (1, n-1)(2, n), (3, n-1)(4, n)\}$ from S_n acting on the code C_1, is a nested 2-PD-set of the minimal size for the code. For $n = 7$, the set $\{Id, (1, n-1)(2, n), (3, n-1)(4, n)(2, 5)\}$ acts in the same way.

   For $n \equiv 2 \pmod 4$, C_1 is self-orthogonal, and doubly even if $n \equiv 2 \pmod 8$.

   The minimum weight of $C_1^\perp$ is 4.

3. *For n even, C_2 is a self-orthogonal $[\binom{n}{3}, \binom{n-2}{2}, 3n-8]_2$ code. For $n \geq 10$ the minimum words are the rows of $A_2 + I$. C_2 is doubly-even if $n \equiv 0 \pmod 4$.*

   For n odd, C_2 is a $[\binom{n}{3}, \binom{n-1}{2}, n-2]_2$ code with a basis of the minimum words $w_{a,b}$.

   The minimum weight of $C_2^\perp$ is 4.

The symmetric group S_n acts transitively of degree $\binom{n}{3}$ as a permutation group on each of these codes.

The proof of these results, together with further results regarding the duals, hulls, the codes generated by the difference of two rows of $A_i + I$, and inter-relationships amongst the codes, can be found in the sections to follow, which are preceeded by a short section giving background definitions and terminology.

In the final section we define the codes W_i and W_Π for general uniform subset graphs $\Gamma(n, k, r)$.

2 Terminology and background

The notation for designs and codes is as in [?]. An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{J})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{J}$ is a t-(v, k, λ) design if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. The design is **symmetric** if it has the same number of points and blocks. The **code $\boldsymbol{C_F(\mathcal{D})}$ of the design** $\mathcal{D}$ over the finite field F is the space spanned by the incidence vectors of the blocks over F. If $\mathcal{Q}$ is any subset of $\mathcal{P}$, then we will denote the **incidence vector** of $\mathcal{Q}$ by $\boldsymbol{v^{\mathcal{Q}}}$, and if $\mathcal{Q} = \{P\}$ where $P \in \mathcal{P}$, then we will write v^P instead of $v^{\{P\}}$. Thus $C_F(\mathcal{D}) = \langle v^B \mid B \in \mathcal{B} \rangle$, and is a subspace of $F^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to F. For any $w \in F^{\mathcal{P}}$ and $P \in \mathcal{P}$, $\boldsymbol{w(P)}$ denotes the value of w at P. If $F = \mathbb{F}_p$ then the **p-rank** of the design, written $\mathrm{rank}_{\boldsymbol{p}}\boldsymbol{(\mathcal{D})}$, is the dimension of its code $C_F(\mathcal{D})$; for $F = \mathbb{F}_p$ we usually write $C_p(\mathcal{D})$ for $C_F(\mathcal{D})$.

All the codes here are **linear codes**, and the notation $[n, k, d]_q$ will be used for a q-ary code C of length n, dimension k, and minimum weight d, where the **weight $\mathrm{wt}(v)$** of a vector v is the number of non-zero coordinate entries. Vectors in a code are also called **words**. The **support**, $\mathrm{Supp}(v)$, of a vector v is the set of coordinate positions where the entry in v is non-zero. So $|\mathrm{Supp}(v)| = \mathrm{wt}(v)$. The **distance $\mathbf{d}(\boldsymbol{u}, \boldsymbol{v})$** between two vectors u, v is the number of coordinate positions in which they differ, i.e., $\mathrm{wt}(u-v)$. A **generator matrix** for C is a $k \times n$ matrix made up of a basis for C, and the **dual** code $C^{\perp}$ is the orthogonal under the standard inner product $(,)$, i.e. $C^{\perp} = \{v \in F^n \mid (v, c) = 0 \text{ for all } c \in C\}$. The **hull** of a code C is the self-orthogonal code $C \cap C^{\perp}$. A **check matrix** for C is a generator matrix for $C^{\perp}$. The **all-one vector** will be denoted by $\jmath$, and is the vector with all entries equal to 1. If we need to specify the length **m** of the all-one vector, we write $\jmath_{\mathbf{m}}$. We call two linear codes **isomorphic** (or permutation isomorphic) if they can be obtained from one another by permuting the coordinate positions. An **automorphism** of a code C is an isomorphism from C to C. The automorphism group will be denoted by $\mathrm{Aut}(C)$, also called the permutation group of C, and denoted by $\mathrm{PAut}(C)$ in [**?**]. Any code is isomorphic to a code with generator matrix in so-called **standard form**, i.e. the form $[I_k \mid A]$; a check matrix then is given by $[-A^T \mid I_{n-k}]$. The set of the first k coordinates in the standard form is called an **information set** for the code, and the set of the last $n-k$ coordinates is the corresponding **check set**.

The **graphs**, $\Gamma = (V, E)$ with vertex set V and edge set E, discussed here are undirected with no loops, apart from the case where **all** loops are included, in which case the graph is called **reflexive**. If $x, y \in V$ and x and y are adjacent, we write $x \sim y$, and xy or $[x, y]$ for the **edge** in E that they define. The **set of neighbours** of $x \in V$ is denoted by $N(x)$, and the **valency of** x is $|N(x)|$. Γ is **regular** if all the vertices have the same valency. A **path** of length r from vertex x to vertex y is a sequence x_i, for $0 \leq i \leq r-1$, of distinct vertices with $x = x_0$, $y = x_{r-1}$, and $x_{i-1} \sim x_i$ for $1 \leq i \leq r-1$. It is **closed** of length r if $x \sim Y$, in which case we write it $(x_0, \ldots, x_{r-1})$. The graph is **connected** if there is a path between any two vertices; $d(x, y)$ denotes the length of the shortest path from x to y.

An **adjacency matrix** A is a $|V| \times |V|$ symmetric matrix with entries a_{ij} such that $a_{ij} = 1$ if vertices x_i and x_j are adjacent, and $a_{ij} = 0$ otherwise. The **neighbourhood design** of Γ is the symmetric 1-$(|V|, k, k)$ design formed by taking the points to be the vertices of the graph and the blocks to be the sets of neighbours of a vertex, for each vertex, i.e. an adjacency matrix as an incidence matrix for the design. If $\Gamma = (V, E)$ is a graph with adjacency matrix A then $A + I_{|V|}$ is an adjacency matrix for the reflexive graph from Γ.

The **code** of Γ over a finite field F is the row span of an adjacency matrix A over the field F, denoted by $C_F(\Gamma)$ or $C_F(A)$. The dimension of the code is the rank of the matrix over F, also written $\mathrm{rank}_p(A)$ if $F = \mathbb{F}_p$, in which case we will speak of the **p-rank** of A or Γ, and write $C_p(\Gamma)$ or $C_p(A)$ for the code. It is also the code over $\mathbb{F}_p$ of the neighbourhood design.

The **uniform subset graph** $\Gamma(n, k, r)$ has for vertices the set of all subsets of size k of a set of size n with two k-subsets x and y defined to be adjacent if $|x \cap y| = r$. The symmetric group S_n always acts on $\Gamma(n, k, r)$, transitively on vertices and edges.

Permutation decoding was first developed by MacWilliams [**?**] and involves finding a set of automorphisms of a code called a PD-set. The method is described fully in MacWilliams and Sloane [**?**, Chapter 16, p. 513] and Huffman [**?**, Section 8]. In [**?**] and [**?**] the definition of PD-sets was extended to that of s-PD-sets for s-error-correction:

Definition 1. *If C is a t-error-correcting code with information set $\mathcal{I}$ and check set $\mathcal{C}$, then a* **PD-set** *for C is a set $\mathcal{S}$ of automorphisms of C which is such that every t-set of coordinate positions is moved by at least one member of $\mathcal{S}$ into the check positions $\mathcal{C}$.*

*For $s \leq t$ an s-***PD-set** *is a set $\mathcal{S}$ of automorphisms of C which is such that every s-set of coordinate positions is moved by at least one member of $\mathcal{S}$ into $\mathcal{C}$.*

The algorithm for permutation decoding is as follows: we have a t-error-correcting $[n, k, d]_q$ code C with check matrix H in standard form. Thus the generator matrix $G = [I_k | A]$ and $H = [-A^T | I_{n-k}]$, for some A, and the first k coordinate positions correspond to the information symbols. Any vector v of length k is encoded as vG. Suppose x is sent and y is received and at most t errors occur. Let $S = \{g_1, \ldots, g_s\}$ be the PD-set. Compute the syndromes $H(yg_i)^T$ for $i = 1, \ldots, s$ until an i is found such that the weight of this vector is t or less. Compute the codeword c that has the same information symbols as yg_i and decode y as cg_i^{-1}.

Notice that this algorithm actually uses the PD-set as a sequence. Thus it is expedient to index the elements of the set S by the set $\{1, 2, \ldots, |S|\}$ so that elements that will correct a small number of errors occur first.

Thus if **nested s-PD-sets** are found for all $1 < s \leq t$ then we can order S as follows: find an s-PD-set S_s for each $0 \leq s \leq t$ such that $S_0 \subset S_1 \ldots \subset S_t$ and arrange the PD-set S as a sequence in this order:

$$S = [S_0, (S_1 - S_0), (S_2 - S_1), \ldots, (S_t - S_{t-1})].$$

(Usually one takes $S_0 = \{id\}$.)

There is a bound on the minimum size that a PD-set S may have, due to Gordon [**?**], from a formula due to Schönheim [**?**], and quoted and proved in [**?**]:

Result 1. *If S is a PD-set for a t-error-correcting $[n,k,d]_q$ code C, and $r = n-k$, then*

$$|S| \geq \left\lceil \frac{n}{r} \left\lceil \frac{n-1}{r-1} \left\lceil \ldots \left\lceil \frac{n-t+1}{r-t+1} \right\rceil \ldots \right\rceil \right\rceil \right\rceil. \tag{1}$$

This result can be adapted to s-PD-sets for $s \leq t$ by replacing t by s in the formula.

3 The graphs Γ_n^i

All codes here are linear and binary, with all spans being over the field $\mathbb{F}_2$.

In this section we establish some general relationships amongst the codes from the matrices $A_i + I$.

Let $\Gamma_n^i = \Gamma(n,3,i) = (V, E^i)$ denote the uniform subset graph with V the set of 3-subsets of $\Omega = \{1, \ldots, n\}$, with $i = 0,1,2$, and A_i an adjacency matrix. Consider the the code $C_2(A_i + I) = C_{A_i+I}$ for $i = 0,1,2$ (in the notation of [**?**]). We use the results of [**?**] concerning the codes $C_2(A_i) = C_{A_i}$. We denote the row of $A_i + I$ corresponding to the vertex x by s_x^i, and the row of A_i corresponding to the vertex x by r_x^i, so $s_x^i = r_x^i + v^x$. The neighbours of x in Γ_n^i are denoted by $N_i(x)$. This graph that includes a loop at each vertex is called a reflexive graph and we will denote it here by $\mathcal{R}\Gamma_n^i$.

From [**?**, Proposition 2.2], for each of the i,

$$C_{A_i}^{\perp} \subseteq C_{A_{i+I}}, \tag{2}$$

and some of the properties of the $C_{A_{i+I}}$ can be deduced from results in [**?**]; we will use those results when we can.

Notation: In the following we will write $C_i = C_2(A_i + I) = C_{A_i+I}$, $i = 0,1,2$.

Also from [**?**, Lemma 2.2], $\jmath \in C_i$ for each i, and it is clear that

$$\jmath = s_x^0 + s_x^1 + s_x^2. \tag{3}$$

Write $x \stackrel{i}{\sim} y$ for x adjacent to y in Γ_n^i, $i = 0,1,2$. The valency ν_i for Γ_n^i is given by:

$$\nu_0 = \binom{n-3}{3}; \;\; \nu_1 = 3\binom{n-3}{2}; \;\; \nu_2 = 3(n-3). \tag{4}$$

The neighbourhood designs of these three graphs and their reflexive associates, respectively, will be denoted by $\mathcal{N}_n^i$ (a 1-$(\binom{n}{3}, \nu_i, \nu_i)$ design) and $\mathcal{R}_n^i$ (a 1-$(\binom{n}{3}, \nu_i + 1, \nu_i + 1)$ design), respectively. The automorphism groups of the graphs, designs and codes always contains S_n, but may be larger in some cases: see Examples **??**, **??**. Using design terminology, we may refer to the vertices x as points and we will denote the block of the design $\mathcal{R}_n^i$ determined by $x = \{a,b,c\}$ and its neighbours by:

$$\bar{x}^i = \overline{\{a,b,c\}}^i = \{x\} \cup N_i(x), \tag{5}$$

so that $\bar{x}^i = \mathrm{Supp}(s_x^i)$.

The codes W_1, W_2 and W_Π defined below have a role to play in the codes from A_i and those from $A_i + I$. In fact for uniform subset graphs on k-sets, similarly defined codes W_i for $1 \leq i \leq k-1$ arise: see [**?**]. In addition, these codes can be defined over any field $\mathbb{F}_p$ and will be similarly related to the codes from the uniform subset graphs. In [**?**, **?**] they are defined as ternary codes.

Definition 2. *For $a, b \in \Omega$, $a \neq b$, let*

$$w_a = \sum_{b,c \in \Omega \setminus \{a\}} v^{\{a,b,c\}},\ W_1 = \langle w_a \mid a \in \Omega \rangle,\ W_1^* = \langle w_a - w_b \mid a, b \in \Omega \rangle; \tag{6}$$

$$w_{a,b} = \sum_{c \in \Omega \setminus \{a,b\}} v^{\{a,b,c\}},\ W_2 = \langle w_{a,b} \mid a, b \in \Omega \rangle,\ W_2^* = \langle w_{a,b} - w_{c,d} \mid a, b, c, d \in \Omega \rangle. \tag{7}$$

Then $\mathrm{wt}(w_a) = \binom{n-1}{2}$, *and* $\mathrm{wt}(w_{a,b}) = n - 2$.

From [**?**, Lemma 4]:

Result 2. *The set $\{w_a \mid a \in \Omega\}$ is a linearly independent set in $\mathbb{F}_2^{|V|}$, and* $\dim(W_1) = n$.

Lemma 1. *For all $n \geq 7$, W_2 has dimension $\binom{n-1}{2}$ and, for any fixed $a \in \Omega$, $\{w_{c,d} \mid c, d \neq a\}$ is a basis for W_2. If n is even $W_2 = W_2^*$; if n is odd W_2^* is a $[\binom{n}{3}, \binom{n-1}{2} - 1, 2n-6]_2$ code.*

Proof: For any n, $\jmath = \sum_{i,j} w_{i,j}$, so $\jmath \in W_2$. Also $\sum_{i=2}^n w_{1,i} = \sum_{i=2}^n \sum_{j \in \Omega \setminus \{1,i\}} v^{\{1,i,j\}} = 0$, so $W_2 = \langle w_{a,b} \mid a, b \in \Omega \setminus \{n\} \rangle$, and if n is even then $w_{1,n}$ is a sum of an even number of $w_{1,i}$, so $W_2 = W_2^*$. To show that $\{w_{a,b} \mid a, b \neq n\}$ is a linearly independent set, suppose $w = \sum_{a,b \in \Omega \setminus \{n\}} \alpha_{a,b} w_{a,b} = 0$. Then the coordinate entry at $\{a, b, n\}$ is $\alpha_{a,b}$, so $\alpha_{a,b} = 0$ for all a, b, and the $\binom{n-1}{2}$ generators are linearly independent.

If n is odd, then $\mathrm{wt}(w_{a,b}) = n - 2$ is odd, so $(w_{a,b}, \jmath) = 1$ for all a, b, and thus $\jmath \in (W_2^*)^\perp$, but $\jmath \notin (W_2)^\perp$. That the minumum weight is $2n - 6$ follows from [**?**], Proposition 6, and the proof of that proposition, and more, in Section 4 of that paper, since the proof holds for any prime p. Then computations with Magma deal with $n \leq 23$. ■

The following definition is given in a way that can apply to the codes over any characteristic. The word w_π is the word $w(\pi)$ of [**?**, Equation 8].

Definition 3. *Let $\Delta = \{a_1, a_2, b_1, b_2, c_1, c_2\} \subset \Omega$ of size 6, and π the partition $[[a_1, a_2], [b_1, b_2], [c_1, c_2]]$ of Δ. Let*

$$X = \{\{a_1, b_1, c_1\}, \{a_1, b_2, c_2\}, \{a_2, b_1, c_2\}, \{a_2, b_2, c_1\}\}$$

and X^c the set of their complements in Δ. Define the weight-8 vector

$$w_\pi = w([a_1, a_2, b_1, b_2, c_1, c_2]) = \sum_{x \in X} v^x - \sum_{x \in X^c} v^x, \tag{8}$$

and

$$W_\Pi = \langle w_\pi \mid \pi \text{ partition of } \Delta \subset \Omega, |\Delta| = 6 \rangle.$$

Lemma 2. *For all $n \geq 7$, $W_\Pi \subseteq C_i^\perp, W_j^\perp$ for $i = 0, 1, 2$, $j = 1, 2$.*

Proof: Immediate. ■

Lemma 3. *For $n \geq 7$,* $\dim(W_\Pi) = \binom{n}{3} - \binom{n}{2}$.

Proof: From [**?**, Lemma 10], $\dim(W_\Pi) \geq \binom{n}{3} - \binom{n}{2}$. We show that $\dim(W_\Pi^\perp) \geq \binom{n}{2}$. From Lemma **??**, $W_1, W_2 \subseteq W_\Pi^\perp$. Let $S = \{w_{a,b} \mid a, b \neq n\} \cup \{w_a \mid a \neq n\}$. This is a linearly independent set: let $w = \sum_{a,b \neq n} \alpha_{a,b} w_{a,b} + \sum_{a \neq n} \beta_a w_a = 0$. Then for all $x, y, z \neq n$,

$$w(\{x, y, z\}) = \alpha_{x,y} + \alpha_{x,z} + \alpha_{y,z} + \beta_x + \beta_y + \beta_z = 0$$

and $w(\{x, y, n\}) = \alpha_{x,y} + \beta_x + \beta_y = 0$. This gives $\alpha_{x,z} + \alpha_{y,z} + \beta_z = 0$, and thus $\alpha_{y,z} = \beta_x$, from which it follows that $\alpha_{x,y} = \beta_z = c$, a constant, for all x, y, z and thus the words in S are linearly independent. Thus $3c = 0$ and so $c = 0$. Thus $\dim(W_\Pi^\perp) \geq |S| = \binom{n}{2}$ and the result follows. ■

Note: 1. A basis for W_Π is thus given in [**?**, Lemma 10], since a linearly independent set is given there. It follows that the result also shows an information set for W_Π. This shows that the dimension of W_Π over any field $\mathbb{F}_p$ with $p \neq 3$, is as given, since the proof that $\dim(W_\Pi) \geq \binom{n}{3} - \binom{n}{2}$ does not depend on the characteristic of the field. For $p = 3$ equality is proved separately in [**?**].

For $i = 0, 1, 2$, let

$$E_i = \langle s_x^i + s_y^i \mid x, y \in V \rangle. \tag{9}$$

Notice that the E_i are all even-weight codes.

Lemma 4. *For $n \geq 7$, $\jmath \in E_i$ for $i = 0, 1, 2$ if and only if one of the following holds:*

$$n \equiv 0 \pmod 4, i = 0, 1;\ n \equiv 1 \pmod 4, i = 0, 2;\ n \equiv 2 \pmod 4, i = 0, 1, 2.$$

Proof: For $n \equiv 3 \pmod 4$, all the $\nu_i + 1$ and $|V|$ are odd, so $\jmath$ is the sum of all the $\binom{n}{3}$ s_x^i, which is an odd number, so $\jmath \notin E_i$ for any i in this case.

For $n \equiv 0 \pmod 4$, $\nu_i + 1$ is odd for $i = 0, 1$, and $|V|$ is even, so $\jmath$ is the sum of all the rows in each case, and this is an even number, so $\jmath \in E_i$ for $i = 0, 1$. If $\jmath \in E_2$, then $\jmath \in \mathrm{Hull}(C_{A_0+1})$ by Lemma **??** (2), which is impossible since $\nu_0 + 1$ is odd. So $\jmath \notin E_2$.

For $n \equiv 1 \pmod 4$, $\nu_i + 1$ is odd for $i = 0, 2$, and $|V|$ is even, so $\jmath$ is the sum of all the rows in each case, and this is an even number, so $\jmath \in E_i$ for $i = 0, 2$. Also, $\jmath = \sum_{a \in \Omega} w_a$ (by Proposition **??**) so it is a sum of an odd number of w_a's and thus from the proof of that proposition, the sum of an odd number of s_x^1's, and thus $\jmath \notin E_1$.

For $n \equiv 2 \pmod 4$, from Lemma **??**, $E_1, E_2 \subseteq E_0$. Suppose $\jmath \notin E_1$. Then $C(A_1 + I) = \langle \jmath, E_1 \rangle$ and $s_x^1 = \jmath + u$ where $u \in E_1$. Noting that $\jmath \in C(A_i + I)^\perp$ for all i since all the $\nu_i + 1$ are even, we see that $(s_x^1, s_y^0) = (\jmath, s_y^0) + (u, s_y^0) \equiv 0 \pmod 2$ which is a contradiction since this is $\equiv 1 \pmod 2$ by Lemma **??** (3). Thus $\jmath \in E_1$, and a similar argument gives $\jmath \in E_2$. Thus also $\jmath \in E_0$. ■

Now we look at the containments amongst the C_i, using ideas and methods from [**?**], but also previous work from [**?**] on the binary codes from the adjacency matrices of the Johnson graphs.

In [**?**], for each $x \in V$, the word $w_x = \sum_{y \overset{i}{\sim} x} r_y$ was defined and used to find relations amongst the C_{A_i}. Here we define, for $x \in V$,

$$w_x^i = \sum_{y \in N_i(x) \cup \{x\}} s_y^i = w_x + v^x, \tag{10}$$

where we add the superscript i to indicate the various Γ_n^i. We can use the results in [**?**] to obtain the following table for the w_x^i, for any fixed $x \in V$, recalling that for any vector w, $w(y)$ denotes it value at the coordinate position y.

- $i = 0$, $w_x^0(y)$;
  1. $y = x$, $w_x^0(x) = \mathrm{wt}(s_x^0) = 1 + \binom{n-3}{3}$, at the one point x;
  2. $|x \cap y| = 2$, $w_x^0(y) = \binom{n-4}{3}$, and there are $3(n-3)$ such points;
  3. $|x \cap y| = 1$, $w_x^0(y) = \binom{n-5}{3}$, and there are $3\binom{n-3}{2}$ such points;
  4. $|x \cap y| = 0$, $w_x^0(y) = \binom{n-6}{3} + 2$, and there are $\binom{n-3}{3}$ such points.
- $i = 1$, $w_x^1(y)$;
  1. $y = x$, $w_x^1(x) = \mathrm{wt}(s_x^1) = 1 + 3\binom{n-3}{2}$, at the one point x;
  2. $|x \cap y| = 2$, $w_x^1(y) = 2\binom{n-4}{2} + (n-4)$, and there are $3(n-3)$ such points;
  3. $|x \cap y| = 1$, $w_x^1(y) = \binom{n-5}{2} + 4(n-5) + 2$, and there are $3\binom{n-3}{2}$ such points;
  4. $|x \cap y| = 0$, $w_x^1(y) = 9(n-6)$, and there are $\binom{n-3}{3}$ such points.

- $i = 2$, $w_x^2(y)$;
  1. $y = x$, $w_x^2(x) = \mathrm{wt}(s_x^2) = 1 + 3(n-3)$, at the one point x;
  2. $|x \cap y| = 2$, $w_x^2(y) = n$, and there are $3(n-3)$ such points;
  3. $|x \cap y| = 1$, $w_x^2(y) = 4$, and there are $3\binom{n-3}{2}$ such points;
  4. $|x \cap y| = 0$, $w_x^2(y) = 0$, and there are $\binom{n-3}{3}$ such points.

As a direct consequence of the observations above for w_x we have:

Lemma 5. *With notation as defined above, $x \in V$,*

1. $n \equiv 0 \pmod 4$*: $w_x^0 = s_x^1$, so $C_1 \subseteq C_0$, $C_2 \subseteq C_0$; $w_x^1 = s_x^1$; $w_x^2 = 0$.*
2. $n \equiv 1 \pmod 4$*: $w_x^0 = s_x^0$; $w_x^1 = s_x^0 + s_x^2 = \jmath + s_x^1$; $w_x^2 = s_x^2$.*
3. $n \equiv 2 \pmod 4$*: $w_x^0 = w_x^1 = w_x^2 = 0$.*
4. $n \equiv 3 \pmod 4$*: $w_x^0 = s_x^2$, so $C_2 \subseteq C_0$, and $C_1 \subseteq C_0$; $w_x^1 = \jmath$; $w_x^2 = s_x^2$.*

Proof: Follows directly from the observations, and that $\jmath = s_x^0 + s_x^1 + s_x^2 \in C_i$ for all i, so if $C_1 \subseteq C_0$, then also $C_2 \subseteq C_0$, and conversely. ■ .

Lemma 6. *For $n \equiv 0, 1 \pmod 4$, for all $x \in V$, $s_x^1 = \sum_{y \in \bar{x}^1} s_y^0$, and $s_x^2 = \sum_{y \not\overset{1}{\sim} x} s_y^0$. Further, $C_1 \subseteq E_0$ for $n \equiv 1 \pmod 4$ and $C_2 \subseteq E_0$ for $n \equiv 0 \pmod 4$.*

Proof: Let $w = \sum_{y \in \bar{x}^1} s_y^0$, and let $x = \{1, 2, 3\}$. First show that $\mathrm{Supp}(w) \supseteq \bar{x}^1$. Clearly $x \in \bar{x}^0$, but $x \notin \bar{y}^0$ for $y \overset{1}{\sim} x$. If $y = \{1, a, b\}$, where $a, b \neq 2, 3$, then $y \in \bar{z}^0$ for $z = \{2, c, d\}$ and $\{3, c, d\}$ for $c, d \neq 1, 2, 3, a, b$, so for $2\binom{n-5}{2}$ verticies $z \in \bar{x}^1$, which cancel. But $y \in \mathrm{Supp}(s_y^0)$ also, so $y \in \mathrm{Supp}(w)$. Similarly for $y = \{2, a, b\}, \{3, a, b\}$. Thus $\mathrm{Supp}(w) \supseteq \bar{x}^1$.

Now suppose $z \in \mathrm{Supp}(w)$ but $z \notin \bar{x}^1$. Then $|z \cap x| = 0, 2$. If $z = \{a, b, c\}$, where $a, b, c \notin \{1, 2, 3\}$, then $z \in \bar{x}^0$ and $z \in \bar{y}^0$ for $y = \{i, d, e\}$ where $i = 1, 2, 3$ and $d, e \notin \{1, 2, 3, a, b, c\}$. Thus it occurs $1 + 3\binom{n-6}{2} \equiv 0 \pmod 2$ times if $n \equiv 0, 1 \pmod 4$. If $|x \cap y| = 2$, suppose $y = \{1, 2, a\}$ where $a \neq 3$. Then $y \in \bar{z}^0$ for $z = \{3, c, d\}$, where $c, d \notin \{1, 2, 3, a\}$. Thus it occurs $\binom{n-4}{2} \equiv 0 \pmod 2$ times for $n \equiv 0, 1 \pmod 4$. Thus we must have $w = s_x^1$.

Then $s_x^2 = \sum_{y \not\overset{1}{\sim} x} s_y^0$ follows from Equation **??**. That $C_1 \subseteq E_0$ for $n \equiv 1 \pmod 4$ and $C_2 \subseteq E_0$ for $n \equiv 0 \pmod 4$ follows by checking the parity of the sums. ■

Corollary 7. *For $n \geq 7$ and $n \equiv 0, 1, 3 \pmod 4$, $C_1 \subseteq C_0$ and $C_2 \subseteq C_0$. Further, $C_0^\perp = C_1^\perp \cap C_2^\perp$.*

Proof: The first statement follows from Lemmas **??** and **??**. Then clearly $C_0^\perp \subseteq C_1^\perp \cap C_2^\perp$. If $w \in C_1^\perp \cap C_2^\perp$, then since $\jmath \in C_i$ for all i and $s_x^0 = \jmath + s_x^1 + s_x^2$, it is clear that $w \in C_0^\perp$ and we have equality. ■

Lemma 8. *For all $n \geq 7$, for all $i = 0, 1, 2$, $j = 1, 2$, $C_i^\perp, W_j^\perp$ have minimum weight at least 4, and exactly 4 for $i, j = 1, 2$. $C_0^\perp$ has minimum weight at most 8. Further*

1. *$C_1^\perp$ has words of weight 4 with support of each of the forms*

$$\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\} \ (\mathit{Type}\ 1) \tag{11}$$
$$\{1,2,4\},\{1,2,5\},\{4,3,6\},\{5,3,6\} \ (\mathit{Type}\ 2) \tag{12}$$
$$\{1,2,3\},\{1,5,6\},\{4,2,6\},\{4,3,5\} \ (\mathit{Type}\ 3), \tag{13}$$

all of which meet a weight-8 word w_π in four points. They are the only weight-4 words in $C_1^\perp$, and they are not in $C_0^\perp$ nor in $C_2^\perp$. There are $15\binom{n}{5} = \frac{90}{n-5}\binom{n}{6}$ of the first kind, $45\binom{n}{6}$ of the second, and $30\binom{n}{6}$ of the third. Also, $C_1^\perp$ is a $[\binom{n}{3}, \binom{n}{3} - n, 4]_2$ code.

2. *$C_2^\perp, W_2^\perp$ have minimum words of weight 4 with support of the form*

$$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}, \tag{14}$$

These weight-4 words are not in $C_0^\perp$ nor in $C_1^\perp$. All words of weight 4 in $C_2^\perp$ have this form.

3. *$C_2^\perp, W_2^\perp$ have words of weight 6 with support of the form*

$$\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{2,3,4\},\{2,3,5\}. \tag{15}$$

The set of supports of such words form the blocks of a 1-$(\binom{n}{3}, 6, 3(n-3)(n-4))$ design such that two points of V are together on 0, 4, or $3(n-4)$ blocks. These weight-6 words are not in $C_0^\perp$. For $n \neq 8$ these are the only words of weight 6 in $C_2^\perp$.

Proof: Since $W_\Pi \subseteq C_i^\perp$ for all i, the minimum weight of any of the $C_i^\perp$ is at most 8. Further, since $\jmath \in C_i$ for all the i, $C_i^\perp$ is an even-weight code, and a simple argument eliminates the possibility of a weight-2 word.

It can be verified directly that the words with supports as shown are in the duals as asserted, and that the three types are the only weight-4 words in $C_1^\perp$ follows by simply considering the possibilities.

That the words with support of the form Equation (**??**) are the only weight-4 words in $C_2^\perp$ follows from an easy argument. For the words of weight 6 of the form Equation (**??**) in $C_2^\perp$, each 5-set from Ω gives $\binom{5}{2} = 10$ words, so there are $10\binom{n}{5}$ in all. This gives $r = 3(n-3)(n-4)$ for the replication number. If $x, y \in V$ have $|x \cap y| = 0$, they will be on no such blocks together; if $|x \cap y| = 1$ they will be on four such blocks (from the 5-set $x \cup y$); if $|x \cap y| = 2$ then they will be on $3(n-4)$ blocks.

For other words of weight 6 in $C_2^\perp$, if n is odd then we can argue as in the proof of [**?**, Proposition 3] that these are the only words of weight 6 in $C_2^\perp$. For n even, if $n = 8$ then $w_{a,b} \in C_2^\perp$ and has weight 6, but for $n = 10, 12$, Magma shows that these are the only words of weight 6 in $C_2^\perp$. For $n \geq 14$ we argue as follows: let $w \in C_2^\perp$ of weight 6, $S = \mathrm{Supp}(w) = \{x_i \mid 1 \leq i \leq 6\}$ and $\Lambda = \bigcup_{i=1}^6 x_i$. Notice that if we suppose that w is not of the form of Equation (**??**) then w can meet any weight-4 vector u in $C_2^\perp$ in at most two points, since if it met in three points then $w + u$ would be a weight-4 vector and hence of the form Equation (**??**), showing that w is of the form of Equation (**??**).

Suppose $\{1,2,3\} \in S$. If there is an element $a \in \Omega \setminus \Lambda$ then $\overline{\{1,2,a\}}^2, \overline{\{1,3,a\}}^2, \overline{\{2,3,a\}}^2$ all meet S so must meet again, in points $\{1,2,b\}, \{1,3,c\}, \{2,3,d\}$ where $b, c, d \in \Lambda$ are all distinct, due to our comment about weight-4 vectors. Thus also $\overline{\{1,b,a\}}^2, \overline{\{3,c,a\}}^2, \overline{\{2,d,a\}}^2$ must meet S again, and in a further three points, which is impossible.

Since $|\Lambda| \leq 18$, if $n > 18$ there is such an $a \in \Omega$ so we have the result. If $n = 18$ it is clear that $\Lambda = \Omega$ is impossible, and for $n = 16, 14$ the argument is similar. We can use Magma for $n = 12, 10$. ∎

Note: 1. The Type 1 and 2 weight-4 words in Lemma **??** are given in [**?**] by $u(\Delta^*)$ where $\Delta^* = [1,2,4,5,3]$, and $w(\Delta^*)$ where $\Delta^* = [1,2,4,5,3,6]$, respectively. The word of weight 4 in $C_2^\perp$ is $w(i,j,k,l)$ of [**?**, Equation (6)].
2. The weight-6 words can be obtained from $w(1,2,3,4) + w(1,2,3,5)$ of Equation (**??**).

The following lemma is proved for the ternary codes in [**?**, Propositions 4,5], and the result holds, with the proof virtually the same, for the binary case. Thus we omit the proof.

Lemma 9. *For any $n \geq 7$, any code $C \subseteq \mathbb{F}_2^V$ with $W_\Pi \subseteq C^\perp$, has minimum weight at least $n-2$ and the words of weight $n-2$ are the $w_{a,b}$ if $n \geq 8$. This is true for $C = W_2$.*

The minimum weight of C_i is at least $n-2$ for $i = 0,1,2$ and all $n \geq 7$.

Note: For $n = 7$, the dual of the code spanned by the weight-8 vectors in Lemma **??** has words of weight $n-2 = 5$ other than the $w_{a,b}$; this code is C_0 and for $n = 7$, $\mathrm{wt}(s_x^0) = 5$.

Lemma 10. *For all $n \geq 7$, $C_1 \cap C_2 = W_1 \cap W_2 = \langle \jmath \rangle$.*

Proof: It is shown in Proposition **??** that $W_1 = C_1$. Also, the argument we use applies to W_2 as well. Then $C_1 \cap C_2 = \langle \jmath \rangle$ if and only if $C_1^\perp + C_2^\perp = \langle \jmath \rangle^\perp = E$, the even-weight subcode of $\mathbb{F}_2^V$. We show that every weight-2 vector is in $C_1^\perp + C_2^\perp$. If $u \in C_1^\perp$ and $v \in C_2^\perp$ denote the vectors with supports of Equation (**??**) and Equation (**??**) respectively, then $u+v = v^{\{2,3,4\}} + v^{\{2,3,5\}} \in C_1^\perp + C_2^\perp$. Thus for any two triples, x, y, meeting in two elements of Ω, $v^x + v^y \in C_1^\perp + C_2^\perp$. Thus $v^{\{1,2,3\}} + v^{\{1,2,4\}} + v^{\{1,2,4\}} + v^{\{1,4,5\}} = v^{\{1,2,3\}} + v^{\{1,4,5\}} \in C_1^\perp + C_2^\perp$, i.e. sums of triples meeting in one point are also included, and thus also $v^{\{1,2,3\}} + v^{\{1,4,5\}} + v^{\{1,4,5\}} + v^{\{4,5,6\}} = v^{\{1,2,3\}} + v^{\{4,5,6\}} \in C_1^\perp + C_2^\perp$. Since W_2 contains these weight-4 and weight-6 words as well, the second statement is also proved. ∎

The following mod 2 values of the inner products of the rows of the $A_i + I$ can be verified directly. In each case, the identity holds for all $x, y \in V$.

Lemma 11. *For $n \geq 7$, any $x, y \in V$ where $\Gamma_n^i = (V, E^i)$,*

1. *For all n, $(s_x^1, s_y^2) = 1$.*
2. *For all even n, $(s_x^0, s_y^2) = 1$ and $(s_x^2, s_y^2) = 0$, $\implies E_2 \subseteq \mathrm{Hull}(C_0)$.*
3. *For $n \equiv 2 \pmod 4$, $(s_x^0, s_y^0) = (s_x^1, s_y^1) = 0$ and $(s_x^0, s_y^1) = 1$.*
4. *For $n \equiv 3 \pmod 4$, $(s_x^0, s_y^1) = (s_x^1, s_y^1) = 1$, $\implies E_1 \subseteq \mathrm{Hull}(C_0)$.*
5. *For $n \equiv 1 \pmod 4$, $(s_x^0, r_x^0) = 0$.*

We can use Lemma **??** to obtain the following lemma:

Lemma 12. *For all $n \geq 7$, $E_i \neq C_{A_i+1}$, and $E_1, E_2 \subseteq E_0$.*

Proof: For the first statement, note that E_i is an even weight code since $\mathrm{wt}(s_x^i + s_y^i) = 2\mathrm{wt}(s_x^i) - 2|s_x^i \cap s_y^i|$. Thus if $\nu_i + 1$ is odd, the claim is immediately true. This covers $n \equiv 3 \pmod 4$ for then all the $\nu_i + 1$ are odd.

The other cases can be deduced from the list above. Thus if $E_1 = C_1$ then, since $s_x^1 + s_y^1 \in C_2^\perp$, we would have also $s_x^1 \in C_2^\perp$, which is false. Similarly $E_2 \neq C_2$.

The same follows for E_0 for n even, so we are left with $n \equiv 1 \pmod 4$. But in this case $\nu_0 + 1$ is odd, so the argument for $n \equiv 3 \pmod 4$ holds here for E_0.

For the second statement, for $n \equiv 0, 1, 3 \pmod 4$ this follows from Corollary **??**. For $n \equiv 2 \pmod 4$, we first show that $\sum_{1,2 \notin x} s_x^0 + \jmath = w_1 + w_2$. Notice that $w_1 + w_2 = \sum_{a,b \geq 3} v^{\{1,a,b\}} + \sum_{a,b \geq 3} v^{\{2,a,b\}}$ since the terms with $\{1, 2, a\}$ cancel. For $x = \{a, b, c\} \not\ni 1, 2$, $x \in \mathrm{Supp}(s_y^0)$ for $y \not\ni 1, 2$, for $\binom{n-5}{3}$ vertices y, and thus it occurs $1 + 1 + \binom{n-5}{3}$ (including s_x^0 and $\jmath$) times, i.e. $\equiv 0 \pmod 2$ times. Also $\{1, 2, a\}$ occurs in $\jmath$ and in s_y^0 for $\binom{n-3}{3}$ vertices y, and thus it does not appear in the left hand side. Vertices $\{1, a, b\}$, where $a, b \geq 3$, occur in $\jmath$ and in $\mathrm{Supp}(s_y^0)$ for $\binom{n-4}{3} \equiv 0 \pmod 2$ vertices y. Similarly for $\{2, a, b\}$, thus proving equality.

Now from Equation (**??**), $s_{\{1,2,3\}}^1 = w_1 + w_2 + w_3$ so $s_{\{1,2,3\}}^1 + s_{\{1,2,4\}}^1 = w_3 + w_4 \in C_0$. It follows easily that $s_x^1 + s_y^1 \in C_0$ for any $x, y \in V$, so $E_1 \subseteq C_0$. Since $\jmath = s_x^0 + s_x^1 + s_x^2$, for any x, y, $s_x^2 + s_y^2 = s_x^0 + s_y^0 + s_x^1 + s_y^1 \in C_0$, and thus $E_2 \subseteq C_0$.

To show they are in E_0, suppose that $w \in E_1$ but $w \notin E_0$. Then $w = \sum_{y \in J} s_y^0$ where $|J|$ is odd. Recall that E_1 is self orthogonal (see Lemma **??**) so $(w, s_x^1) = 0$ for all $x \in V$. But by Lemma **??** (3), $(s_x^0, s_y^1) = 1$ for any x, y, so $(w, s_x^1) = |J| \equiv 1 \pmod 2$. Thus $E_1 \subseteq E_0$. A similar argument holds for E_2 since by Lemma **??** (2), $(s_x^0, s_y^2) = 1$. ∎

Lemma 13. *For $n \equiv 2 \pmod 4$, $A_0^2 = A_1^2 = I$ and hence C_0 and C_1 are self-orthogonal, and doubly-even if $n \equiv 2 \pmod 8$. For n even C_2 is self-orthogonal, and doubly-even if $n \equiv 0 \pmod 4$.*

Proof: First notice that if $A^2 = I$ then $(A + I)(A + I) = 0$, so C_{A+I} is self-orthogonal. We will show that $(r_x^i, r_y^i) = \delta_{x,y}$ for $i = 0, 1$ when $n \equiv 2 \pmod 4$ and for $i = 2$ and any even n.

Take first $i = 0$: $(r_x^0, r_x^0) = \binom{n-3}{3} \equiv 1 \pmod 2$. If $x \overset{0}{\sim} y$ then x, y have $\binom{n-6}{3} \equiv 0 \pmod 2$ common neighbours in Γ_n^0. If $|x \cap y| = 1$, they will have $\binom{n-5}{3} \equiv 0 \pmod 2$ common neighbours. If $|x \cap y| = 2$, they will have $\binom{n-4}{3} \equiv 0 \pmod 2$ common neighbours. Thus $A_0^2 = I$. Since $\nu_0 + 1 = \binom{n-3}{3} + 1 \equiv 0 \pmod 4$ for $n \equiv 2 \pmod 8$, we see that C_{A_0+1} is doubly-even in this case.

For $i = 1$: $(r_x^1, r_x^1) = 3\binom{n-3}{2} \equiv 1 \pmod 2$. If $x \overset{1}{\sim} y$ then x, y have $\binom{n-5}{2} + 4(n-5) \equiv 0 \pmod 2$ common neighbours in Γ_n^1. If $|x \cap y| = 0$, they will have $9(n-6) \equiv 0 \pmod 2$ common neighbours. If $|x \cap y| = 2$, they will have $(n-4) + 2\binom{n-4}{2} \equiv 0 \pmod 2$ common neighbours. Thus $A_1^2 = I$. Since $\nu_1 + 1 = 3\binom{n-3}{2} + 1 \equiv 0 \pmod 4$ for $n \equiv 2 \pmod 8$, we see that C_{A_1+1} is doubly-even in this case.

For $i = 2$ and n even: $(r_x^2, r_x^2) = 3(n-3) \equiv 1 \pmod 2$. If $x \overset{2}{\sim} y$ then x, y have $2 + (n-4) \equiv 0 \pmod 2$ common neighbours in Γ_n^2. If $|x \cap y| = 0$, they have no common neighbours. If $|x \cap y| = 1$, they will have $4 \equiv 0 \pmod 2$ common neighbours. Thus $A_2^2 = I$. Since $\nu_2 + 1 = 3(n-3) + 1 \equiv 0 \pmod 4$ for $n \equiv 0 \pmod 4$, we see that C_{A_2+1} is doubly-even for $n \equiv 0 \pmod 4$. ■

4 The codes C_1

We look first at the codes C_1 since more can be proved about this case than about C_0 or C_2.

Proposition 14. *For $n \geq 7$, C_1 is a $[\binom{n}{3}, n, \binom{n-1}{2}]_2$ code, and $C_1 = W_1$. Its weight distribution is given by $\binom{n}{r}$ words of weight $n_r = r\binom{n-r}{2} + \binom{r}{3}$ from the sum of r distinct w_a, for each $0 \leq r \leq n$. For $n \geq 8$, the minimum words are the w_a. For $n = 7$ there are another 21 words from $r = 5$. $C_1^\perp$ has minimum weight 4.*

The set $\mathcal{I} = \{\{i, n-1, n\} \mid 1 \leq i \leq n-2\} \cup \{\{n-3, n-2, n-1\}, \{n-3, n-2, n\}\}$ is an information set for C_1 for all $n \geq 7$. For $n \geq 8$, with this information set, the sequence of automorphisms $\{Id, (1, n-1)(2, n), (3, n-1)(4, n)\}$ from S_n acting on the code C_1, is a nested 2-PD-set of the minimal size for the code. For $n = 7$, the set $\{Id, (1, n-1)(2, n), (3, n-1)(4, n)(2, 5)\}$ acts in the same way.

For $n \equiv 0 \pmod 4$, $C_1 = C_{A_1}^\perp = C_{A_0}^\perp$.

Proof: We first note that it is easy to prove that, for $x = \{a, b, c\} \in V$, and for all n,

$$w_a + w_b + w_c = s_x^1 = s_{\{a,b,c\}}^1, \tag{16}$$

and hence that $w_a = s_{\{a,b,c\}}^1 + s_{\{a,b,d\}}^1 + s_{\{a,c,d\}}^1$ for any distinct $a, b, c, d \in \Omega$. Thus $C_1 = W_1$, so $\dim(C_1) = n$. It is also easy to see that the set of n rows s_x^1 with $x \in \mathcal{I}$ can be put into row echelon form.

For the weight distribution we follow the same reasoning as in [?, Lemma 6]. Thus let $\Delta = \{a_1, \ldots, a_r\} \subseteq \Omega$, where $0 \leq r \leq n$, and let

$$w = \sum_{i=1}^{r} w_{a_i} = \sum_{i=1}^{r} \sum_{x,y \neq a_i} v^{\{a_i,x,y\}} = \sum_{i=1}^{r} \Big(\sum_{x,y \in \Omega \setminus \Delta} v^{\{a_i,x,y\}} + \sum_{x \in \Omega \setminus \Delta, j \neq i} v^{\{a_i,a_j,x\}} + \sum_{j,k \neq i} v^{\{a_i,a_j,a_k\}} \Big),$$

of weight $n_r = r\binom{n-r}{2} + \binom{r}{3}$. This is for each of the $\binom{n}{r}$ choices of Δ.

The smallest weight occurs for $r = 1$, and gives the n words w_a of weight $\binom{n-1}{2}$: that this is the smallest weight follows because $n_1 < n_r$ where $r \geq 2$ simplifies to $4r^2 + r(4-6n) + (3n^2 - 9n + 6) > 0$. The discriminant of this quadratic in r is $-12n^2 + 96n - 80$ and this is negative for $n \geq 8$, so $n_r > n_1$ for all $r \geq 2$ for $n \geq 8$, and so the w_a are all the minimum words. For $n = 7$, $n_5 = n_1$ so an additional 21 words occur. For $r = n$, $\jmath = \sum_{a \in \Omega} w_a$.

That the given sets are information sets for C_1 when $n \equiv 3 \pmod 4$ can be verified directly, as can the 2-PD-sets for $n = 7$ and for $n \geq 8$.

Since $C_{A_1}^\perp \subseteq C_1$ and for $n \equiv 0 \pmod 4$ they have the same dimension, by [?], they are equal, and equal to $C_{A_0}^\perp$, by [?].

That $C_1^\perp$ has minumum weight 4 was shown in Lemma ??. ■

Note: 1. The n_r are not necessarily distinct for distinct r, but each value of r gives $\binom{n}{r}$ of that weight. For example, for $n = 7$, $n_2 = n_6 = 20$, and there are $\binom{7}{2} + \binom{7}{6} = 28$ words of this weight. However, it can be verified that $n_r = n_s$ cannot have solutions for $n \equiv 0 \pmod 4$ so in that case there are exactly n distinct non-zero weights.
2. Using the Vandermonde identity, it can be verified that $n_r + n_{n-r} = \binom{n}{3}$, so if $n_r = n_s$, then $n_{n-r} = n_{n-s}$.

Proposition 15. *For $n \geq 11$, $E_1 = W_1^*$ is a $[\binom{n}{3}, n-1, (n-2)(n-3)]_2$ code and the minimum words are the words $w_a + w_b$, for $a \neq b$, $a, b \in \Omega$. For $7 \leq n \leq 10$, E_1 is as follows:*

- *$n = 7$, a $[35, 6, 16]_2$ code, minimum words the 35 words of weight $n_4 = 16$ from $s_x^1 + s_y^1$ where $x \overset{1}{\sim} y$;*
- *$n = 8$, a $[56, 7, 26]_2$ code, minimum words the 28 words of weight $n_6 = 26$ from $s_x^1 + s_y^1$ where $|x \cap y| = \emptyset$;*
- *$n = 9$, a $[84, 8, 38]_2$ code, minimum words the 84 words of weight $n_6 = 38$ from $s_x^1 + s_y^1$ where $|x \cap y| = \emptyset$;*
- *$n = 10$, a $[120, 9, 56]_2$ code, minimum words the $\binom{10}{2} + \binom{10}{6} = 255$ words of weight $n_2 = n_6 = 56$ from $s_x^1 + s_y^1$ where $x \overset{1}{\not\sim} y$.*

*The set $\mathcal{I} \setminus \{\{n-2, n-1, n\}\}$ from Proposition **??** is an information set for E_1, and the set of automorphisms given there is a nested 2-PD-set of the minimal size for the code.*

For $n \equiv 1 \pmod 4$, $E_1 = C_{A_1}^{\perp}$.

Proof: Recall that E_1 is an even-weight code spanned by $s_x^1 + s_y^1$ and thus also by even sums of the w_a. Thus $E_1 = W_1^*$. So we take r to be even. Note that $n_2 = (n-2)(n-3)$. Also we show in Lemma **??** that for all $n \geq 7$, $E_i \neq C_{A_i+1}$, so $\dim(E_1) = n - 1$.

For fixed n and r, where $n \geq 7$ and $n \geq r \geq 3$ let $P(n, r)$ be the statement $n_2 < n_r$, i.e.

$$(n-2)(n-3) < r\binom{n-r}{2} + \binom{r}{3}.$$

We show that if $n \geq 11$ then $P(n, r)$ is true for all $3 \leq r \leq n$. The statement $P(n, r)$ is equivalent to

$$0 < 3n^2(r-2) + 3n(10 - r(2r+1)) + 2r(r^2+1) - 36 = p(n, r).$$

It is easy to see that $P(n, n)$ is true for all n and $P(n, n-1)$ is true for $n \geq 8$. The discriminant of $p(n, r)$ is less than 0 if

$$9(10 - r(2r+1))^2 < 24(r-2)(r(2r^2+1) - 18)$$

i.e.

$$3(2r+5)^2(r-2)^2 < 8(r-2)^2(2r^2 + 4r + 9).$$

Solving shows this holds for $r \geq 8$, so for $n, r \geq 8$, $P(n, r)$ is true. We need to consider $3 \leq r \leq 7$ and for this we consider $p(n, r)$ for these values. Direct computation yields that $p(n, 3) > 0$ for $n \geq 8$; $p(n, 4) > 0$ for $n \geq 9$; $p(n, 5) > 0$ for $n \geq 9$; $p(n, 6) > 0$ for $n \geq 11$; $p(n, 7) > 0$ for $n \geq 11$. Thus we have $p(n, r) > 0$ for $n \geq 11$ and all $r \geq 3$, as required.

For the $7 \leq n \leq 11$, direct computation with the n_r and the corresponding weights gives the result.

That the given sets are information sets for E_1 can be verified directly, as can the 2-PD-set.

The final statement follows since by [**?**], for $n \equiv 1 \pmod 4$, $C_{A_1}^{\perp}$ is spanned by the $w_a + w_b$. ■

In fact, from [**?**], we can deduce the following for C_1 and E_1, giving s-PD-sets for s up to $\lceil \frac{n(n-1)}{6} \rceil - 1$. With notation and information sets as in Propositions **??** and **??**:

Result 3. *For $n \geq 8$, taking the following elements of S_n in their natural action on triples of elements of $\Omega = \{1, 2, \ldots, n\}$:*

$$\Sigma_1 = \{(n,i) \mid 1 \leq i \leq n-2\} \cup \{Id\};\ \Sigma_2 = \{(n-1,i) \mid 1 \leq i \leq n-2\} \cup \{Id\};$$
$$\Sigma_3 = \{(n-2,i) \mid 1 \leq i \leq n-4\} \cup \{Id\};\ \Sigma_4 = \{(n-3,i) \mid 1 \leq i \leq n-4\} \cup \{Id\},$$

where Id is the identity element of S_n, let $\Sigma_{1,2} = \Sigma_1\Sigma_2 \setminus \{(n,a)(n-1,a) \mid 1 \leq a \leq n-2\}$ and $\Sigma_{3,4} = \Sigma_3\Sigma_4 \setminus \{(n-2,a)(n-3,a) \mid 1 \leq a \leq n-4\}$. Then $\Sigma = \Sigma_{1,2}\Sigma_{3,4}$ is an s-PD-set of size $n^4 - 10n^3 + 37n^2 - 60n + 39$ for C_1 for $s \leq \lceil \frac{n^2}{6} \rceil - 1$, and for E_1 for $s \leq \lceil \frac{n(n-1)}{6} \rceil - 1$.

Note: 1.This result is stated and proved in [**?**] for the codes $C_{A_0}^{\perp}$ for $n \equiv 0 \pmod 4$ and $C_{A_1}^{\perp}$ for $n \equiv 1 \pmod 4$ respectively. These are identified as being the codes spanned by the words w_a and $w_a + w_b$, respectively, and are thus our codes C_1 and E_1, respectively, for any $n \geq 7$.
2. The sets given in the result are large, and are not nested, but they do correct a lot of errors. Smaller sets can be constructed if s is given some small value, as in Propositions **??** and **??** where $s = 2$.

Corollary 16. *For $n \geq 7$, $\mathrm{Hull}(C_1)$ is: $\{0\}$ if $n \equiv 0 \pmod 4$; $\langle \jmath \rangle$ if $n \equiv 1 \pmod 4$; C_1 if $n \equiv 2 \pmod 4$; E_1 if $n \equiv 3 \pmod 4$, and doubly-even.*

Proof: Let $C = C_1$. We have $(w_a, w_a) = \binom{n-1}{2}$, and $(w_a, w_b) = n - 2$ for $a \neq b$.
(i) $n \equiv 0 \pmod 4$: Then $(w_a, w_a) = 1$ and $(w_a, w_b) = 0$ for $a \neq b$. If $w = \sum_{i=1}^{r} w_{a_i} \in \mathrm{Hull}(C)$, then $(w_{a_i}, w) = 1$ unless $w = 0$. Thus $\mathrm{Hull}(C) = \{0\}$.
(ii) $n \equiv 1 \pmod 4$: Then $(w_a, w_a) = 0$ and $(w_a, w_b) = 1$. If $w = \sum_{i=1}^{r} w_{a_i} \in \mathrm{Hull}(C)$, then $(w_a, w) = r$ if $a \notin \{a_1, \ldots, a_r\}$ and $(w_{a_i}, w) = r - 1$. This is consistent only if $w = \jmath$, since then $n - 1 = 0$. So $\mathrm{Hull}(C) = \langle \jmath \rangle$.
(iii) $n \equiv 2 \pmod 4$: C is self-orthogonal by Lemma **??**, so $\mathrm{Hull}(C) = C$.
(iv) $n \equiv 3 \pmod 4$: Here $(w_a, w_a) = (w_a, w_b) = 1$ for all a, b, so $w_a + w_b \in C^{\perp}$. It is clear that $w_a \notin C^{\perp}$, so $\mathrm{Hull}(C) = \langle w_a + w_b \mid a, b \in \Omega \rangle = \langle s_x^1 + s_y^1 \mid x, y \in V \rangle = E_1$.

For the final statement, note that for $n \equiv 3 \pmod 4$, $\mathrm{Hull}(C_1) = E_1$ is spanned by the $w_a + w_b$ of weight $(n-2)(n-3) \equiv 0 \pmod 4$ and since it is self-orthogonal, this proves the statement. ■

5 The codes C_2

For $i = 2$, the graph $\Gamma_n^2 = J(n, 3)$, a Johnson graph $J(n, k)$, and the 2-ranks of the adjacency matrices A and $A + I$ are given in [**?**], while the codes from A are studied in [**?**]. Further results for $k = 3$ are in [**?**]. We sum up what conclusions we have for the codes $C_{A_2+I} = C_2$ from these results, using in particular [**?**, Proposition 1].

Result 4. *For $n \geq 7$, $\Gamma_n^2 = (V, E) = J(n, 3)$, A_2 an adjacency matrix:*

1. *for n even, $C_{A_2} = \mathbb{F}_2^{|V|}$, C_2 is self-orthogonal of 2-rank $\binom{n-2}{2}$;*
2. *for n odd, $C_2 = C_{A_2}^{\perp} = W_2$ is a $[\binom{n}{3}, \binom{n-1}{2}, n-2]_2$ code with minimum words the $w_{a,b}$, for $a, b \in \Omega$. $C_2^{\perp}$ is a $[\binom{n}{3}, \binom{n-1}{3}, 4]_2$ code spanned by words of weight-4 of the form of Equation (**??**). Further, $\mathrm{Hull}(C_2) = 0 = \mathrm{Hull}(W_2)$.*

   For any $a \in \Omega$, a basis for C_2 is $\{w_{b,c} \mid b, c \in \Omega \setminus \{a\}\}$.

Note: For any n, $s_{\{a,b,c\}}^2 = w_{a,b} + w_{a,c} + w_{b,c}$ (see [**?**, p.175], but note there is a typographical error in the relevant equation), so $C_2 \subseteq W_2$. For n odd, $w_{a,b} = \sum_{c \in \Omega \setminus \{a,b\}} s_{\{a,b,c\}}^2$, so $C_2 = W_2$. For n even $\sum_{c \in \Omega \setminus \{a,b\}} s_{\{a,b,c\}}^2 = 0$.

Corollary 17. *For $n \geq 7$, odd, $\mathrm{Hull}(W_2^*) = \{0\}$ if $n \equiv 3 \pmod 4$, and $\mathrm{Hull}(W_2^*) = \langle \jmath \rangle$ if $n \equiv 1 \pmod 4$.*

Proof: Since $\jmath = \sum_{\{a,b\}} w_{a,b}$, a sum of $\binom{n}{2}$ of the $w_{a,b}$, we have $\jmath \in W_2^*$ if $n \equiv 1 \pmod 4$. For all n odd, $\jmath \in (W_2^*)^\perp$. If $n \equiv 1 \pmod 4$ then $\jmath \in \mathrm{Hull}(W_2^*)$. If $w \in \mathrm{Hull}(W_2^*)$ and $w \neq 0$, then, since by Result **??**, $\mathrm{Hull}(W_2) = \{0\}$, $w \notin W_2^\perp$, so $(w, w_{a,b}) = 1$ for all $a, b \in \Omega$. This holds for any $u \in \mathrm{Hull}(W_2)$, and so $(u + w, w_{a,b}) = 0$, so $u = w$ and $\mathrm{Hull}(W_2^*)$ has at most one non-zero element. Thus for $n \equiv 1 \pmod 4$, $\mathrm{Hull}(W_2^*) = \langle \jmath \rangle$. If $n \equiv 3 \pmod 4$ then $\jmath \notin W_2^*$, but $\jmath + w_{a,b} \in W_2^*$. For $w \in \mathrm{Hull}(W_2^*)$, $(w, \jmath + w_{a,b}) = 0 = (w, \jmath) + (w, w_{a,b}) = (w, w_{a,b})$. Thus $w = 0$, completing the proof. ■

Proposition 18. *For $n \geq 8$ even, $C_2 \subsetneq W_2$ is a self-orthogonal $[\binom{n}{3}, \binom{n-2}{2}, 3n-8]_2$ code. For $n \geq 10$ the words of minimum weight are the rows s_x^2.*

If S is the set of 2-subsets of $\Omega \setminus \{1, n\}$, then the $\binom{n-2}{2}$ rows s_z^2 with $z \in \{\{1, a, b\} \mid \{a, b\} \in S\}$ form a basis for C_2. Furthermore, $w_{a,b} \in C_2^\perp$ for all $a, b \in \Omega$, and $C_2^\perp$ has minimum weight 4.

Proof: We have the dimension and self-orthogonality from Result **??**, and that $C_2 \subset W_2$ was noted above. Write $w(1,2,3,4)$ for the word in $C_2^\perp$ with support given in Equation (**??**). It is easy to check that $w_{a,b} \in C_2^\perp$ for all a, b. Let $w \in C_2$ and suppose that $\{1,2,3\} \in \mathrm{Supp}(w)$. Then $w(1,2,3,a)$ must meet w again, once or three times, for each $a \in \Omega \setminus \{1,2,3\}$. Let $S = \mathrm{Supp}(w)$. Suppose that for

- $A = \{a_i \mid i \in \{1, \ldots, l\}\}$ we have $w(1,2,3,a_i) \subset S$;
- $X = \{x_i \mid i \in \{1, \ldots, r\}\}$ we have $\{1, 2, x_i\} \in S$;
- $Y = \{y_i \mid i \in \{1, \ldots, s\}\}$ we have $\{1, 3, y_i\} \in S$;
- $Z = \{z_i \mid i \in \{1, \ldots, t\}\}$ we have $\{2, 3, z_i\} \in S$,

and suppose that $0 \leq r \leq s \leq t$. So $n - 3 = r + s + t + l$, and

$$|S| \geq 1 + (r + s + t) + 3l = (3n - 8) - 2(r + s + t).$$

Now use the fact that $(w, w_{a,b}) = 0$ for all $a, b \in \Omega$. Thus for

- $\{a, b\} = \{1, x\}, \{1, y\}$, $x \in X$, $y \in Y$, need at least $\lceil \frac{r+s}{2} \rceil$ more elements S_X, of the form $\{1, x, y\}$ or $\{1, x_i, x_j\}$ *etc.*, in S;
- $\{a, b\} = \{2, x\}, \{2, z\}$, $x \in X$, $z \in Z$, need at least $\lceil \frac{r+t}{2} \rceil$ more elements S_Y, of the form $\{2, x, z\}$ or $\{2, x_i, x_j\}$ *etc.*, in S;
- $\{a, b\} = \{3, y\}, \{3, z\}$, $x \in X$, $y \in Y$, need at least $\lceil \frac{s+t}{2} \rceil$ more elements S_Z, of the form $\{3, y, z\}$ or $\{3, z_i, z_j\}$ *etc.*, in S.

Thus we need at least another $|S_X| + |S_Y| + |S_Z| \geq r + s + t$ elements, implying that $|S| \geq 3n - 8 - (r + s + t)$.

Now we resort once again to the weight-4 words of the form $w(1, x, y, a)$, $w(2, x, z, a)$, $w(3, y, z, a)$, where $x \in X, y \in Y, z \in Z, a \in A$ requiring that such words must meet w evenly. Thus

- $w(1, x, y, b) \ni \{1, x, y\}$, for $b \in A \cup Z$, meets w again, and there is no overlap for distinct pairs of the form $\{x, y\}$, so we need at least another $\lceil \frac{r+s}{2} \rceil (t + l)$ elements;
- $w(2, x, z, b) \ni \{2, x, z\}$, for $b \in A \cup Y$, meets w again, and there is no overlap for distinct pairs of the form $\{x, z\}$, so we need at least another $\lceil \frac{r+t}{2} \rceil (s + l)$ elements;
- $w(3, y, z, b) \ni \{3, y, z\}$, for $b \in A \cup X$, meets w again, and there is no overlap for distinct pairs of the form $\{y, z\}$, so we need at least another $\lceil \frac{s+t}{2} \rceil (r + l)$ elements.

Supposing that $r \le s \le t$, then $r+s+t \le 3t$ and if $\frac{r+s}{2}(t+l) > 3t$ we would need more than $r+s+t$ extra elements in S; this is certainly the case if $r+s>6$. Thus we need consider the cases where $r+s \le 6$. All the cases give weight at least $3n-8$ with equality only if $r=s=t=0$ if $n>8$, so that $w = s^2_{\{1,2,3\}}$. For $n=8$ there are further words of weight 16 from the difference of two rows of A_2+I that have $n=8$ common non-zero entries, for example $s^2_{\{1,2,3\}} + s^2_{\{1,2,4\}}$. We leave the details to the reader.

For the basis, we set up an ordering of the elements in V that will be useful when we consider bases for C_0. Thus we order the set S in some fixed way, and order the rows s^2_z with $z \in \{\{1,a,b\} \mid \{a,b\} \in S\}$ in the same way. Now for the columns, the first block of $\binom{n-2}{2}$ will correspond to the vertices $\{\{n,a,b\} \mid \{a,b\} \in S\}$. This set of columns is labelled $\mathcal{C}_1$. The next set of columns, labelled $\mathcal{C}_2$ will correspond to the $\binom{n-2}{2}$ vertices $\{\{1,a,b\} \mid \{a,b\} \in S\}$. The next, labelled $\mathcal{C}_3$ will be the $\binom{n-2}{3}$ vertices of 3-sets on $\Omega \setminus \{1,n\}$. Finally we take for $\mathcal{C}_4$ the $n-2$ vertices $\{\{1,i,n\} \mid 2 \le i \le n-1\}$.

Now it is clear from this labelling that the matrix of the s^2_z for $z \in \{\{1,a,b\} \mid \{a,b\} \in S\}$ has the identity on the left-hand side, and since $\binom{n-2}{2}$ is the dimension of C_2, we do have a basis.

That $C_2^\perp$ has minimum weight 4 follows from Lemma **??**. ■

Lemma 19. *For $n \ge 7$, for E_2:*

1. *if n is even, the words $s^2_x + s^2_y$ for $x \overset{2}{\sim} y$, have weight $4(n-4)$ and there are $3\binom{n}{4}$ of them;*
2. *if n is odd, $E_2 = W_2^*$, of minimum weight $2(n-3)$ from the words $w_{a,b} + w_{a,c}$.*

Proof: 1. For n even, the weight can be checked directly; that there are $3\binom{n}{4}$ of them follows from the observation that for any 4-subset $\{a,b,c,d\}$ of Ω,

$$s^2_{\{a,b,c\}} + s^2_{\{a,b,d\}} + s^2_{\{a,c,d\}} + s^2_{\{b,c,d\}} = 0.$$

2. For n odd, clearly $E_2 = W_2^*$ and the minimum weight was noted in Lemma **??**. That the weight of $w_{a,b} + w_{a,c}$ is as stated can be verified directly. There are $3\binom{n}{3}$ such words as for any 3-subset $\{a,b,c\}$ of Ω, we get three distinct words. ■

Note: The minimum weight of E_2 for n even is $4(n-4)$ according to computations with Magma for $8 \le n \le 14$.

6 The codes C_0

From [**?**, Proposition 1] and [**?**], we can deduce the following:

Proposition 20. *For $n \ge 7$ and odd, C_0 is a $[\binom{n}{3}, \binom{n}{2}, n-2]_2$ code, and $C_0 = \langle w_{i,j}, w_i \mid i,j \in \Omega \setminus \{n\}\rangle$, which is a basis. For $n \ge 9$, the words $w_{a,b}$ are the minimum words. $C_0^\perp$ has minimum weight 8 and $C_0^\perp = W_\Pi$.*

If $n \equiv 1 \pmod 4$, $C_0 = C_{A_0}^\perp$ and $\mathrm{Hull}(C_0) = \{0\}$.

If $n \equiv 3 \pmod 4$ then $C_0 = C_2 \oplus \mathrm{Hull}(C_0)$, $\mathrm{Hull}(C_0) = \mathrm{Hull}(C_1)$.

Proof: When n is odd, C_1 and C_2 are subcodes of C_0, by Corollary **??**, so $w_{i,j}, w_i \in C_0$, by Result **??** and Proposition **??**. By Lemma **??**, C_0 has minimum weight at least $n-2$, but since $\mathrm{wt}(w_{a,b}) = n-2$, this is the minimum weight.

By Result **??**, $C_2 = \langle w_{i,j} i,j \in \Omega \setminus \{n\}\rangle$, and by Proposition **??**, $C_1 = \langle w_i \mid i \in \Omega\rangle$. Since $s^0_x = \jmath + s^1_x + s^2_x$, and $\jmath = \sum_{i\in\Omega} w_i \in C_2$, it follows that $C_0 = \langle w_{i,j}, w_i \mid i,j \in \Omega \setminus \{n\}\rangle$. By Lemma **??**, this is a linearly independent set that spans $W_\Pi^\perp$, and so we have the first result, and that $C_0 = W_\Pi^\perp$.

Since $C_0^\perp \subseteq C_{A_0}$, if $n \equiv 1 \pmod 4$ we can use [**?**, Theorem 1 (3)] where it was shown that C_{A_0} is $[\binom{n}{3}, \binom{n}{3} - \binom{n}{2}, 8]_2$, to see that $C_0^\perp = C_{A_0}$, since they have the same dimension. Thus $\mathbb{F}_2^{|V|} = C_0 + C_{A_0} = C_0 + C_0^\perp$, and thus $\mathrm{Hull}(C_0) = \{0\}$.

If $n \equiv 3 \pmod 4$, from Lemma **??** (4) we have $s^1_x + s^1_y \in \mathrm{Hull}(C_0) \cap \mathrm{Hull}(C_1)$ for any x, y, and hence also $w_a + w_b \in \mathrm{Hull}(C_0) \cap \mathrm{Hull}(C_1)$ for any $a, b \in \Omega$. Also recall that $\jmath = \sum_{i=1}^n w_i$. Since $\jmath = s^0_x + s^1_x + s^2_x$ from

Equation (**??**), we have, from Equation (**??**) $s_x^0 = s_x^2 + \sum_{i=1}^n w_i + w_a + w_b + w_c$, where $x = \{a,b,c\}$, and since n is odd, the last sum of $n+3$ w_i is in $\mathrm{Hull}(C_0) \cap \mathrm{Hull}(C_1)$, so $C_0 = C_2 + \mathrm{Hull}(C_0)$, since the right-hand side is in C_0. To show it is a direct sum, we have $C_2 = C_{A_2}^\perp$, from Result **??**, and $C_0^\perp = C_{A_0} = C_{A_2}$, from [**?**, Theorem 1 (4)], so, we have $C_2 \cap C_0^\perp = \{0\}$. Thus also $C_2 \cap \mathrm{Hull}(C_0) = \{0\}$, and $\mathrm{Hull}(C_0) = \mathrm{Hull}(C_1)$ since they have the same dimension.

That for $n > 7$ the words $w_{a,b}$ are precisely the minimum words (as they are for C_2) follows from Lemma **??**. The argument fails for $n = 7$, when there are other words of weight $n - 2 = 5$, *viz.* the s_x^0, as noted earlier.

To show that the minimum weight of $C_0^\perp$ is 8, we can use the fact that $C_0^\perp \subseteq C_2^\perp$, and that the only words of weight 4 and 6 in $C_2^\perp$ for n odd have the form from Equation (**??**) and Equation (**??**), respectively, and that we have shown that these words are not in $C_0^\perp$. Since $C_0^\perp = W_\Pi$, 8 is its minimum weight. ■

Proposition 21. *For $n \geq 8$, $n \equiv 0 \pmod 4$, C_0 is a $[\binom{n}{3}, \binom{n-1}{2} + 1, d]_2$ code where $n - 1 \leq d \leq 3n - 8$ for $n > 8$, $d = 11$ for $n = 8$. A basis for C_0 is the set of $\binom{n-2}{2}$ rows s_z^2 that give a basis for C_2 as in Proposition **??**, together with the words $w_i \in W_1$ for $1 \leq i \leq n-1$. The words $w_{a,b}$ are not in C_0. Further, $C_2 \subsetneq E_0$, and $E_2 = \mathrm{Hull}(C_0) \subsetneq E_0$. The minimum weight of $C_0^\perp$ is 8, and $C_0^\perp = C_1^\perp \cap C_2^\perp$.*

Proof: We arrange the vertices in the columns $\mathcal{C}_i$ as described in Proposition **??**. We take the $\binom{n-2}{2}$ rows that form a basis for C_2 and follow these with the rows showing the words w_i, for $1 \leq i \leq n-1$. We will not need w_n since it can be formed form the w_i for $1 \leq i \leq n-1$, and $\boldsymbol{\jmath} \in C_2$.

So we need to show that these words are linearly independent and span C_0. For this we use the upper $\binom{n-2}{2}$ rows to reduce the entries for the w_i that are in the first $\binom{n-2}{2}$ columns to zero. For w_1 there are no entries there; all its entries are all of $\mathcal{C}_2$ and all of $\mathcal{C}_4$. For w_i, where $2 \leq i \leq n-1$, there are entries at $\{i,j,n\}$ for $j \neq 1, i, n$, i.e. $n-3$ entries, so to remove them we add to w_i the $n-3$ rows s_z^2 for $z = \{1,i,j\}$. This reduces the entries in $\mathcal{C}_1$ to zero. For the columns in $\mathcal{C}_2$, the columns corresponding to $\{1,i,j\}$ will have entries in each of the rows s_z^2 taken, i.e. $n-3$ entries 1. In the row for w_i there are also entries 1 in each of these columns, so these will cancel, leaving 0 here too. For the columns $\{1,j,k\}$ where $j,k \neq i$, there will be precisely two non-zero entries, at the rows corresponding to $\{1,i,j\}$ and $\{1,i,k\}$. When added to w_i below, these will not change the zeros already there to $1's$. Thus $\mathcal{C}_2$ is also all zero. For $\mathcal{C}_3$, again $\{i,j,k\}$ will occur twice, at $\{1,i,j\}$ and at $\{1,i,k\}$, so these will have no effect on the entries in w_i. Finally, for $\mathcal{C}_4$, the single entry 1 at the column $\{1,i,n\}$ will change to zero, while the other entries zero will change to 1.

Thus the part of the matrix corresponding the rows w_2 to w_{n-1} will have zeros up to $\mathcal{C}_3$ where they will correspond to the w_i words on $\Omega \setminus \{1,n\}$, and are thus still linearly independent, regardless of the remaining part in $\mathcal{C}_4$. The row for w_1 has zero in $\mathcal{C}_1$ then all entries in $\mathcal{C}_2$ are 1. Thus it cannot be dependent on the upper words or the w_i for $2 \leq i \leq n-1$. Thus we have $\binom{n-2}{2} + n - 1 = \binom{n-1}{2} + 1$ linearly independent vectors. We need only make sure that they generate C_0. But this is clear since $s_x^0 = \boldsymbol{\jmath} + s_x^1 + s_x^2$, and $s_x^1 \in \langle w_i \mid 1 \leq i \leq n \rangle = \langle \boldsymbol{\jmath}, w_i \mid 1 \leq i \leq n-1 \rangle$ since $w_n = \boldsymbol{\jmath} + \sum_{i=1}^{n-1} w_i$.

Regarding the bounds on the minimum weight, for the upper bound we have $\mathrm{wt}(s_x^2) = 3n - 8$ as an upper bound, except when $n = 8$ and $\mathrm{wt}(s_x^0) = 11 < 3n - 8 = 16$. For the lower bound we use Lemma **??** which shows that the bound is at least $n - 2$ and equal to this only if $w_{a,b} \in C_0$. Thus suppose that $w_{a,b} \in C_0$. From the basis we have just obtained, $w_{a,b} = w + v$ where $w \in C_2$ and $v \in \langle w_i \mid 1 \leq i \leq n-1 \rangle$. Now for all $\Delta \subset \Omega$ of size 4, $(w_{a,b}, w(\Delta)) = 0$, where $w(\Delta)$ is defined in Equation (**??**). Thus $(v, w(\Delta)) = 0$ for all Δ. Suppose $v = \sum_{i \in I} w_i$, where $I \subseteq \Omega \setminus \{n\}$. Notice that $(w_i, w(\Delta)) = 0$ if $i \notin \Delta$, and 1 if $i \in \Delta$. If $|I| \geq 3$, suppose $i,j,k \in I$, $\Delta = \{n,i,j,k\}$; then $(v, w(\Delta)) = 1$, a contradiction. So $1 \leq |I| \leq 2$, and since $n \geq 8$, there exists $j,k,l \notin I$, $i \in I$, so with $\Delta = \{i,j,k,l\}$ we have $(v, w(\Delta)) = 1$, also a contradiction. Thus $w_{a,b} \notin C_0$ which establishes the lower bound.

To show that $s_x^2 \in E_0$ for $n \equiv 0 \pmod 4$, from Lemma **??** 1(a), $w_x^0 = \sum_{y \in N_i(x) \cup \{x\}} s_y^0 = s_x^1 = \boldsymbol{\jmath} + s_x^0 + s_x^2$ by Equation **??**. But $\sum_{x \in V} s_x^0 = \boldsymbol{\jmath}$, so we have $s_x^2 = \boldsymbol{\jmath} + \sum_{y \in N_i(x)} s_y^0 = \sum_{y \notin N_i(x)} s_y^0$ and this is a sum of an even number of s_y^0 for $n \equiv 0 \pmod 4$. So $C_2 \subseteq E_0$.

To show that $E_2 = \mathrm{Hull}(C_0)$, from Lemma **??** #2. we have that $E_2 \subseteq \mathrm{Hull}(C_0)$. Suppose $w \in \mathrm{Hull}(C_0)$. Then $w \in C_0$ implies that $w = \sum_{z \in J} s_z^2 + \sum_{j \in I} w_j$, $I \subseteq \{1, \ldots, n-1\}$, and we also have $(w, s_x^0) = 0$ all $x \in V$.

Now by Lemma **??** #2 we have $(s_z^2, s_x^0) = 1$, so $(\sum_{z\in J} s_z^2, s_x^0) = |J|$ for all x. We show that $(\sum_{j\in I} w_j, s_x^0)$ cannot be a constant over all $x \in V$ unless $I = \emptyset$. First notice that $(w_a, s^0_{\{a,b,c\}}) = 1$, and if $d \notin \{a,b,c\}$, then $(w_d, s^0_{\{a,b,c\}}) = \binom{n-4}{2} \equiv 0 \pmod 2$. Assuming $I \neq \emptyset$, let $i \in I$, and suppose there is a $j \in \Omega$, $j \neq n$ such that $j \notin I$. Then writing $u = \sum_{k\in I} w_k$, $(u, s^0_{\{n,i,j\}}) = 1$. If there exists $k \in I$, $k \neq i$ then $(u, s^0_{\{n,i,k\}}) = 0$, which cannot hold, so $I = \{i\}$ and there is another $k \neq n, j$, such that $k \notin I$. Then $(u, s^0_{\{n,j,k\}}) = 0$, again a contradiction. So the only possibility is $I = \{1, \ldots, n-1\}$. But then $(u, s^0_{\{n,1,2\}}) = 0$, but $(u, s^0_{\{1,2,3\}}) = 1$. Thus $I = \emptyset$, $|J| \equiv 0 \pmod 2$, and $w \in E_2$.

To show that $\mathrm{Hull}(C_0) \subset E_0$, from Lemma **??**, $C_0 = E_0 + \langle s_x^0\rangle$ for any $x \in V$. If w is in the hull but $w \notin E_0$ then $w = u + s_x^0$ where $u \in E_0$. For any $y \in V$, $(s_y^2, u) = 0$ by Lemma **??** #2, so $(w, s_y^2) = 0 + (s_x^0, s_y^2) = 1$. But $C_2 \subseteq C_0$ by Corollary **??**, so $\mathrm{Hull}(C_0) \subseteq C_0^\perp \subseteq C_2^\perp$. Thus we have a contradiction, and deduce that $\mathrm{Hull}(C_0) \subset E_0$.

For the minimum weight of $C_0^\perp$, we know from Lemma **??** that this is at most 8. We have $C_0^\perp = C_1^\perp \cap C_2^\perp \subseteq C_2^\perp$ by Corollary **??**. The only weight-4 words in $C_2^\perp$ have the form of Equation (**??**), and these are not in $C_0^\perp$. Thus $C_0^\perp$ has minimum weight at least 6, and since it must be even, at most 8. But for $n > 8$, by Lemma **??**, the only weight-6 words in $C_2^\perp$ are those from Equation (**??**), and these are not in $C_0^\perp$. For $n = 8$ there are more weight-6 words but have the form $w_{a,b}$ and these are not in $C_0^\perp$. Thus $C_0^\perp$ has minimum weight 8, since $W_\Pi \subseteq C_0^\perp$. ■

Proposition 22. *For $n \geq 10$, $n \equiv 2 \pmod 4$, C_0 is a $[\binom{n}{3}, \binom{n-1}{2}, d]_2$ self-orthogonal code with $n \leq d \leq 4(n-4)$. A basis for C_0 is the set of $\binom{n-2}{2} - 1$ words $s_z^2 + s^2_{\{1,2,3\}}$ where z ranges over the basis for C_2 given in Proposition **??**, excluding $s^2_{\{1,2,3\}}$, together with the words $w_i + w_1$ for $2 \leq i \leq n-1$, and the single word $s^0_{\{1,2,3\}} + \jmath$. The words $w_{a,b}$ are not in C_0.*

Further, $C_0^\perp$ has minimum weight 8.

Proof: This can be proved by showing that a matrix similar to that in Proposition **??** can be put into row echelon form. In this case the rows correspond first to the $\binom{n-2}{2} - 1$ words $s_z^2 + s^2_{\{1,2,3\}}$ for z from the vertices in the basis for C_2 as in Proposition **??**, then the words $w_1 + w_i$ for $2 \leq i \leq n-1$, then finally $s^0_{\{1,2,3\}} + \jmath$. The columns are the same as those from $\mathcal{C}_1$ of Proposition **??**, except that we remove for now the column of $\{2,3,n\}$. Next come the columns labelled $\mathcal{C}_4$ before, followed by the column for $\{2,3,n\}$. The columns $\mathcal{C}_2$ and $\mathcal{C}_4$ are then as before.

It can be shown that this reduces to a row echelon form such that each row has a leading entry and thus that the dimension of the space C spanned by these rows is the number of rows, i.e. $\binom{n-1}{2}$. To show this is C_0, notice first that the sum of the first $\binom{n-2}{2} - 1$ rows is $\jmath$, so $\jmath \in C$, and hence also $s^0_{\{1,2,3\}} \in C$. Since any word $s_y^2 + s_z^2$ can be obtained from the first $\binom{n-2}{2} - 1$ rows, they are all in C. Also we have $w_1 + w_n \in C$, since $\jmath \in C$, so all the sums $s_y^1 + s_z^1$ are in C. Since $\jmath = s_x^0 + s_x^1 + s_x^2$, we have, for any $x \in V$, $s_x^0 + s^0_{\{1,2,3\}} = s_x^1 + s^1_{\{1,2,3\}} + s_x^2 + s^2_{\{1,2,3\}} \in C$, and hence $s_x^0 \in C$ for all x, and $C = C_0$.

For the bounds on the minimum weight, as an upper bound we have $\mathrm{wt}(s_x^2 + s_y^2) = 4n - 16$ when $x \overset{2}{\sim} y$. For the lower bound again we know from Lemma **??** that a word of weight $n-2$ must be of the form $w_{a,b}$. Thus suppose $w_{a,b} \in C_0$. Since C_0 is self-orthogonal, we must have $(w_{a,b}, s_x^0) = 0$ for all $x \in V$. Clearly this is not the case for $x = \{a,b,c\}$, so $w_{a,b} \notin C_0$ which establishes the lower bound, since C_0 is an even-weight code, so $n-1$ is not possible.

For the minimum weight of $C_0^\perp$, we know from Lemma **??** that this is at most 8. Suppose $C_0^\perp$ has a word w of weight 4. It must be orthogonal to all the words in the basis as described above, and hence $(w, s^2_{\{1,a,b\}} + s^2_{\{1,2,3\}}) = (w, w_i + w_1) = 0$ for $\{a,b\}$ any 2-subset of $\Omega \setminus \{1,n\}$ and $2 \leq i \leq n-1$. Thus $(w, s^2_{\{1,a,b\}}) = (w, s^2_{\{1,2,3\}})$ and $(w, w_i) = (w, w_1)$ for this range of a, b, i. Suppose first that $(w, w_i) = 0$, for $1 \leq i \leq n-1$ so that $w \in C_1^\perp$ since clearly this holds for $i = n$ as well. So w meets all the w_i evenly so that all the i present in an element of $\mathrm{Supp}(w)$ occur an even number of times. If an $i \in \Omega$ occurs four times, then we can assume $i = 1$ (since S_n acts transitively on Ω) and it follows easily that $\mathrm{Supp}(w)$ has the form

$\{\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\}\}$ of Type 1 in Lemma **??**. Then $(w, s^2_{\{1,2,3\}}) = 0$ but $(w, s^2_{\{1,2,4\}}) = 1$, which is a contradiction. So we assume that no w_i meet it four times and then it follows that w has the form of Type 2 or Type 3 of the same lemma. Suppose $\text{Supp}(w) = \{\{1,2,4\},\{1,2,5\},\{4,3,6\},\{5,3,6\}\}$. Then $(w, s^2_{\{1,2,3\}}) = 0$ but $(w, s^2_{\{1,5,7\}}) = 1$. So $(w, w_1) = 0$ is impossible. A similar argument rules out the third type of weight-4. Thus we must have $(w, w_i) = 1$ for $1 \le i \le n-1$. Then $\text{Supp}(w)$ meets every i in the range, so if $n > 10$ this is impossible. For $n = 10$ we could have $\text{Supp}(w) = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{8,9,10\}\}$, in which case $(w, s^2_{\{1,2,3\}}) = 1$ but $(w, s^2_{\{1,2,4\}}) = 0$, which is a contradiction. So the minumum weight of $C_0^\perp$ is at least 6.

If $w \in C_0^\perp$ has weight 6, then (w, s^2_x) is a constant over $x \in V$. Since $C_2^\perp$ has only one type of weight-6 word and that is not in $C_0^\perp$, it follows that $(w, s^2_x) = 1$ for all $x \in V$. This means that every $\bar{x}^2$ meets $\text{Supp}(w)$. If $S = \text{Supp}(w) = \{x_i \mid 1 \le i \le 6\}$ and $\Lambda = \bigcup_{i=1}^6 x_i$, then $|\Lambda| \le 18$, so if $n \ge 20$ and $a, b \in \Omega \setminus \Lambda$, then any $x = \{a, b, c\}$ will have $\bar{x}^2$ not meeting S. If $n = 18$ and $|\Lambda| = 17$ or 18 it is easy to see that there are x such that $\bar{x}^2$ does not meet S; similarly for $n = 14$ and $|\Lambda| = 13$ or 14. For $n = 10$ we can easily use Magma to obtain the result. Thus the minimum weight of $C_0^\perp$ is 8, since $W_\Pi \subseteq C_0^\perp$. ∎

For E_0, recall first from Lemma **??** that $E_1, E_2 \subseteq E_0$.

Computational observation 1. *For $n \ge 7$, for E_0:*

1. *for $n = 7$ the minimum weight is 6 and $s^0_x + s^0_y$ for $x \overset{0}{\sim} y$ are minimum words;*
2. *for $n = 8$ the minimum weight is 12 and words of the form $s^2_{\{1,2,3\}} + s^2_{\{1,2,4\}} + w_5 + w_6 + w_7 + w_8$ are minimum words;*
3. *for $n \ge 9$ and $n \equiv 1, 2, 3 \pmod 4$ E_0 has the same minimum weight as E_2 and shares minimum words, of weight $4(n-4)$ for $n \equiv 2 \pmod 4$, and $2(n-3)$ for n odd;*
4. *for $n \ge 12$ and $n \equiv 0 \pmod 4$, the minimum weight is $3n - 8$, from the s^2_x.*

Proof: So far this is only by Magma. ∎

Using Magma [**?**, **?**], we found the following:

Example 1. For $n = 10$, we know that the self-orthogonal code C_0 has dimension 36. From Magma it has minimum weight 24 and there are 5355 minimum words, in three orbits under $\text{Aut}(\Gamma^0_{10}) \cong S_{10}$, and they can be described as follows:

- 630 words of the form $s^2_x + s^2_y$ where $x \overset{2}{\sim} y$;
- 1575 words of the form
$$s^2_{\{1,2,3\}} + s^2_{\{1,2,4\}} + s^0_{\{1,5,6\}} + s^0_{\{2,5,6\}} \text{ and } s^2_{\{1,2,3\}} + s^2_{\{1,2,4\}} + s^0_{\{3,5,6\}} + s^0_{\{4,5,6\}};$$
- 3150 words of the form
$$s^2_{\{1,2,3\}} + s^2_{\{1,2,4\}} + s^2_{\{5,6,9\}} + s^2_{\{7,8,9\}} + s^0_{\{5,6,10\}} + s^0_{\{7,8,10\}}.$$

The supports of the 5355 minimum words form the blocks of a 2-$(120, 24, 207)$ design $\mathcal{D}$ which is such that $\text{Aut}(\mathcal{D}) = \text{Aut}(C_0)$, and this group has order $47377612800 = 2^{16} * 3^5 * 5^2 * 7 * 17$, is simple, 2-transitive on points, primitive on blocks, and is isomorphic to the symplectic group $Sp_8(2)$ (or, alternatively, the simple orthogonal group $SO_9(2)$). Thus the 5355 minimum words are in one orbit under $\text{Aut}(C_0)$. The binary code $C_2(\mathcal{D})$ of the design has dimension 35, and does not contain the words s^0_x, although $\jmath \in C_2(\mathcal{D})$, it being the sum of all the rows. Both $C_0^\perp$ and $C_2(\mathcal{D})^\perp$ have minimum weight 8, containing the words of weight 8 defined by partitions, as in Lemma **??**. That the minimum weight of $C_2(\mathcal{D})^\perp$ is at least 8 follows also from the design parameters, since the replication number r for $\mathcal{D}$ is 1071, so if a word in $C_2(\mathcal{D})^\perp$ has weight s we must have $s - 1 \ge r/\lambda = 1071/207 = 5.2$, so $s \ge 7$ and since $\jmath \in C_2(\mathcal{D})$, it is of even weight and thus $s \ge 8$. The automorphism group of each of the other codes for $n = 10$, and of all the graphs and neighbourhood designs, is just S_{10}.

i	n	$\dim(C_i)$	mw C_i	min wds	mw $C_i^\perp$	$\dim(H_i)$	mw H_i	mw E_i	other
0	0	$\binom{n-1}{2}+1$	$\leq 3n-8$	$s_x^2?*$	8	$\binom{n-2}{2}-1$	$\leq 4(n-4)$	$\leq 3n-8$	$H_0 = E_2$
	1	$\binom{n}{2}$	$n-2$	$w_{a,b}$	8	0	0	$\leq 2(n-3)$	
	2	$\binom{n-1}{2}$	$\leq 4(n-4)$	$s_x^2+s_y^2?$	8	$\binom{n-1}{2}$	$\leq 4(n-4)$	$\leq 4(n-4)$	$H_0 = C_0$
	3	$\binom{n}{2}$	$n-2$	$w_{a,b}^\dagger$	8	$n-1$	$2\binom{n-2}{2}^\dagger$	$\leq 2(n-3)^\dagger$	$H_0 = E_1$
1	0	n	$\binom{n-1}{2}$	w_a	4	0	0	$2\binom{n-2}{2}$	$C_1 \subset C_0$
	1	n	$\binom{n-1}{2}$	w_a	4	1	$\binom{n}{3}$	”	$C_1 \subset C_0$
	2	n	$\binom{n-1}{2}$	w_a	4	n	$\binom{n-1}{2}$	”	$E_1 \subset E_0$
	3	n	$\binom{n-1}{2}$	w_a	4	$n-1$	$2\binom{n-2}{2}$	”	$C_1 \subset C_0$
2	0	$\binom{n-2}{2}$	$3n-8$	s_x^2*	4	$\binom{n-2}{2}$	$3n-8$	$\leq 4(n-4)$	$C_2 \subset E_0$
	1	$\binom{n-1}{2}$	$n-2$	$w_{a,b}$	4	0	0	$2(n-3)$	$C_2 \subset C_0$
	2	$\binom{n-2}{2}$	$3n-8$	s_x^2	4	$\binom{n-2}{2}$	$3n-8$	$\leq 4(n-4)$	$E_2 \subset E_0$
	3	$\binom{n-1}{2}$	$n-2$	$w_{a,b}$	4	0	0	$2(n-3)$	$C_2 \subset C_0$

Table 1: Binary codes of C_i for $n \geq 7$

Note: The design and code acted on by $Sp_8(2)$ is also constructed, in a different way, in [**?**].

Example 2. For $n = 7$, Γ_7^0 is the odd graph $\mathcal{O}_3$, and $\mathrm{Aut}(C_0) = S_8 = \mathrm{Aut}(\mathcal{N}_7^1) = \mathrm{Aut}(\mathcal{R}_7^1) = \mathrm{Aut}(\Gamma_7^1) = \mathrm{Aut}(\mathcal{R}\Gamma_n^1) = \mathrm{Aut}(C_1)$. The other groups are all S_7.

Extra automorphisms that then generate S_8 can easily be defined in this case, as was already described in [**?**]. For $a \in \Omega = \{1, \ldots, 7\}$, let $\Omega_a = \Omega \setminus \{a\}$, and for $x \in V$, let $x^{c_a} = \Omega_a \setminus x$. Then the map α_a defined by

$$\alpha_a : x \mapsto \begin{cases} x & \text{if } a \in x \\ x^{c_a} & \text{if } a \notin x \end{cases}$$

is easily seen to be an automorphism of the graphs $\Gamma_7^1, \mathcal{R}\Gamma_7^1$, and thus of their neighbourhood designs and codes. It was already shown to be an automorphism of the code C_0 in [**?**], since for $n = 7$, Γ_7^0 is an odd graph, $\mathcal{O}_3$. The α_a are not automorphisms of the graph $\Gamma_7^0 = \mathcal{O}_3$. They are also not automorphisms of the Johnson graph Γ_7^2.

7 Conclusion

The table shown is a summary of some of the facts we have established about the codes. The column labelled i refers to the codes from $A_i + I$, for $i = 0, 1, 2$ and $H_i = \mathrm{Hull}(C_i)$. The second column denotes the value of n modulo 4. The entries with $*$ in the first and 9^{th} rows are for $n > 8$; for $n = 8$ the minimum weight of C_{A_0+1} is $11 = |s_x^0| < |s_x^2| = 16$, while the words of weight 16 are not only the rows s_x^2 in C_{A_2+1}. The entries with a $\dagger$ in the fourth row is for $n > 7$; for $n = 7$ there are words of weight $n - 2 = 5$ other than the $w_{a,b}$, and the minimum weight of H_0 is 16: see Proposition **??**, since $H_0 = E_1$. An entry ? means we have not proved this, i.e. this is from Magma. For $n \equiv 2 \pmod 4$, all the C_i are self-orthogonal and thus equal to their own hulls. In the set of rows for $i = 0$, the entry $4(n-4)$ is the weight of $s_x^2 + s_y^2$ when $x \overset{2}{\sim} y$, since in $A_2 + I$, rows meet in 0 points if $|x \cap y| = 0$, 4 points if $|x \cap y| = 1$, and in n points if $x \overset{2}{\sim} y$. The minimum weight for E_1 is for $n \geq 11$; see Proposition **??** for $7 \leq n \leq 10$. Note that the minimum weight of C_i is at least $n - 2$ for all the $i = 0, 1, 2$, by Lemma **??** so we do not include this lower bound for the minimum weight in the table.

In the introduction we mentioned a series of codes W_i, W_Π over any $\mathbb{F}_p$ that can be used to establish results about codes from the uniform subset graphs $\Gamma(n, k, r) = (V, E)$. The W_i are defined in the obvious way: if $x \subseteq \Omega$ and $|x| = i$, then the word $w_x = \sum_{y \in V, x \subset y} v^y$, and $W_i = \langle w_x \mid x \subset \Omega, |x| = i \rangle$, where the span is over $\mathbb{F}_p$.

For the code W_Π we make use of partitions of subsets of size $2k$ of Ω. Let such a partition π be

$$[[a_1, a_2], [b_1, b_2], \ldots, [k_1, k_2]],$$

then the word w_π will have the set of k-sets $\{a_{i_1}, b_{i_2}, \ldots, k_{i_k}\}$ as support with the sign being determined by giving $x = \{a_1, b_1, \ldots, k_1\}$ the sign "+", and then demanding that any other k-set in the support with intersection of size $k-1$ with x will have sign "$-$", and then applying this in general to get the signs on all the 2^k vertices. Alternatively the words can be defined inductively: for example, from the partition for $k=3$ given in Definition **??**, we can get to one for $k=4$ with the extra partition set $[d_1, d_2]$ by adjoining d_1 to all the elements of the sets X and X^c, keeping the same signs, and then do the same with d_2, but switiching the signs. Another interpretation takes the 2^k vertices in the support of w_π as the vertices of the k-cube, $\mathcal{Q}_k$, i.e. the Hamming graph $H(k,2)$, with alternate signs on the vertices. For all $1 \le i \le k-1$, $W_\Pi \subseteq W_i^\perp$.

These words were used in [**?**] in the binary case for codes from Johnson and odd graphs. The codes W_i, W_Π will come into play for all the codes, over any $\mathbb{F}_p$, from the adjacency matrix $A_i, A_i + I, A_i + I + J$ (for complementary graphs).