Codes associated with the odd graphs

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Abstract

Linear codes arising from the row span over any prime field $\mathbb{F}_p$ of the incidence matrices of the odd graphs $\mathcal{O}_k$ for $k \geq 2$ are examined and all the main parameters obtained. A study of the hulls of these codes for $p = 2$ yielded that for $\mathcal{O}_2$ (the Petersen graph), the dual of the binary hull from an incidence matrix is the binary code from points and lines of the projective geometry $PG_3(\mathbb{F}_2)$, which leads to a correspondence between the edges and vertices of $\mathcal{O}_2$ with the points and a collection of ten lines of $PG_3(\mathbb{F}_2)$, consistent with the codes.

The study also gives the dimension, the minimum weight, and the nature of the minimum words, of the binary codes from adjacency matrices of the line graphs $L(\mathcal{O}_k)$.

1 Introduction

Recent investigations of the codes from the $|V| \times |E|$ incidence matrices of k-regular connected graph $\Gamma = (V, E)$ in, for example [8, 23, 14, 20, 21], yielded observations that led to a more general approach for this study, using edge-connectivity of graphs, in [6, 5]. This showed that, under certain very broad conditions on Γ, the codes over any field $\mathbb{F}_p$ from an incidence matrix G have the properties that: the dimension is $|V|$ or $|V| - 1$; the minimum weight is k and the words of weight k are the scalar multiples of the rows of G; there are no words of weight i such that $k < i < 2k - 2$; the words of weight $2k - 2$ are the scalar multiples of the differences of two rows of G corresponding to adjacent vertices. Thus the graph can be retrieved from the code. Such properties are reminiscent of the codes from finite projective planes: see [1, Chapter 6]. Codes from the adjacency matrices of graphs do not behave in such a uniform way. However, since for $p = 2$, $G^T G$ is an adjacency matrix for the line graph of Γ, $L(\Gamma)$, in such cases we can use the facts about the code from the incidence matrix for Γ for information about the binary code from the adjacency matrix of $L(\Gamma)$, including the dimension and minimum weight. In particular, the codes will not be trivial.

In this paper we examine these codes from incidence matrices of the odd graphs $\mathcal{O}_k$, and deduce properties of the binary codes from adjacency matrices of the line graphs $L(\mathcal{O}_k)$ and also the hulls of these codes, where the hull of a code C is $C \cap C^{\perp}$. The odd graphs $\mathcal{O}_k$ for $k \geq 2$ are the uniform subset graphs $\Gamma(2k + 1, k, 0)$ whose vertices are the subsets of size k of a set of size $2k + 1$, with two vertices being adjacent if the two k-subsets intersect in the empty set[1]. They are thus $(k + 1)$-regular graphs. Binary codes from the adjacency matrices of these graphs were examined in [10, Chapter 6]. Here we consider p-ary codes from incidence matrices for these graphs, along with binary codes from the adjacency matrices of their line graphs, and the hulls of these.

Our main results are collected in the following theorem, where we use the notation that if A is a matrix then $C_p(A)$ denotes the row span of A over the prime field $\mathbb{F}_p$:

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[1]Frequently denoted by O_{k+1} in the literature

Theorem 1. *For $k \geq 2$, let $G_k = [g_{i,j}]$ be a $\binom{2k+1}{k} \times \frac{k+1}{2}\binom{2k+1}{k}$ incidence matrix for the odd graph $\mathcal{O}_k$, and let L_k be an adjacency matrix for the line graph $L(\mathcal{O}_k)$. For p any prime, let $\varepsilon_p = 0$ if p is odd, $\varepsilon_2 = 1$. Then:*

1. *For any prime p, $C_p(G_k)$ is a $\left[\frac{k+1}{2}\binom{2k+1}{k}, \binom{2k+1}{k} - \varepsilon_p, k+1\right]_p$ code.*

   If $k \geq 3$, the minimum words are the scalar multiples of the rows of G_k, there are no words of weight i where $k+1 < i < 2k$, and the words of weight $2k$ are the scalar multiples of the differences of two rows corresponding to two adjacent vertices.

   If $p = 2$, the same is true for $k = 2$. For p odd, $C_p(G_2)$ has more words of weight 3.

2. *If $E(G_k) = \langle g_{i,j} - g_{i,m} \mid 1 \leq i \leq 2k+1 \rangle$ over $\mathbb{F}_2$, then $E(G_k) = C_2(L_k)$. If $k = 2^l - 1$ for some $l \geq 2$, then $C_2(L_k) = C_2(G_k)$; otherwise $C_2(L_k)$ has codimension 1 in $C_2(G_k)$ and is a $\left[\frac{k+1}{2}\binom{2k+1}{k}, \binom{2k+1}{k} - 2, 2k\right]_2$ code, with the words of weight $2k$ the rows of L_k.*

3. *For all $k \geq 2$, $\mathrm{Hull}(C_2(G_k))$ and $\mathrm{Hull}(C_2(L_k))$ have minimum weight at least $2k+2$, and either they are equal or one has codimension 1 in the other. For k even, $\dim(\mathrm{Hull}(C_2(G_k))) = \binom{2k-1}{k} + 2^{k-1} - 1$; for k odd, $\dim(\mathrm{Hull}(C_2(G_k))) = \binom{2k}{k-1} - 1$.*

   Further, for the strongly regular $(10, 3, 0, 1)$ Petersen graph $\mathcal{O}_2$,

$$(\mathrm{Hull}(C_2(G_2)))^\perp = C_2(G_2) + C_2(G_2)^\perp = C_2(PG_{3,1}(\mathbb{F}_2)) = \mathcal{H}_4,$$

   where $\mathcal{H}_r$ denotes the Hamming code of length $2^r - 1$, $\langle \mathrm{Hull}(C_2(G_2)), \boldsymbol{\jmath}_{15} \rangle = C_2(PG_{3,2}(\mathbb{F}_2))$. There is a correspondence between the edges and vertices of $\mathcal{O}_2$ and the 15 points and a set of ten lines of $PG_3(\mathbb{F}_2)$, consistent with the codewords. The edges of the 15 8-cycles of $\mathcal{O}_2$ are the supports of the non-zero words of $\mathcal{H}_4^\perp$, with complements the 15 Fano planes $PG_2(\mathbb{F}_2)$ in $PG_3(\mathbb{F}_2)$.

General terminology is given in Section 2. The results collected in the theorem appear as propositions in Sections 3, 4, 5, and 6. Some further general results about binary codes of adjacency matrices of line graphs and their hulls are shown in Sections 4 and 5. The results for the Petersen graph are in Section 6. This is followed by step-by-step procedures to correspond the points of $PG_3(\mathbb{F}_2)$ with the edges of $\mathcal{O}_2$, and the converse operation of obtaining the graph from the points and a set of ten lines of $PG_3(\mathbb{F}_2)$. Section 7 concerns the use of these codes for permutation decoding.

2 Background, terminology, and previous results

2.1 Designs and codes

The notation for designs and codes is as in [1]. An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{J})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{J}$ is a t-(v, k, λ) design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. A design is **symmetric** if it has the same number of points as blocks. The **code $C_F(\mathcal{D})$ of the design** $\mathcal{D}$ over the finite field F is the space spanned by the incidence vectors of the blocks over F. If $\mathcal{Q} \subseteq \mathcal{P}$, then we will denote the **incidence vector** of $\mathcal{Q}$ by $\boldsymbol{v^{\mathcal{Q}}}$. Thus $C_F(\mathcal{D}) = \langle v^B \mid B \in \mathcal{B} \rangle$, and is a subspace of $F^{\mathcal{P}}$. For any $w \in F^{\mathcal{P}}$ and $P \in \mathcal{P}$, $\boldsymbol{w(P)}$ denotes the value of w at P. If $F = \mathbb{F}_p$ then the **p-rank** of $\mathcal{D}$, written $\mathrm{rank}_p(\mathcal{D})$, is the dimension of $C_p(\mathcal{D})$, writing $C_p(\mathcal{D})$ for $C_F(\mathcal{D})$.

All the codes here are **linear codes**, and the notation $[n, k, d]_q$ is used for a q-ary code C of length n, dimension k, and minimum weight d, where the **weight $\mathbf{wt(v)}$** of a vector v is the number of non-zero coordinate entries. The **support**, $\mathrm{Supp}(v)$, of a vector v is the set of coordinate positions where the entry in v is non-zero. A **generator matrix** for C is a $k \times n$ matrix with rows a basis for C, and the **dual** code $C^\perp$ is the orthogonal under the standard inner product $(,)$, i.e. $C^\perp = \{v \in F^n \mid (v, c) = 0 \text{ for all } c \in C\}$. If $C = C_p(\mathcal{D})$, where $\mathcal{D}$ is a design, then $C \cap C^\perp$ is the **hull** of $\mathcal{D}$ or C. A **check matrix** for C is a generator matrix for $C^\perp$. The **all-one vector** will be denoted by $\boldsymbol{\jmath}$, and is the vector with all entries

equal to 1. The all-one vector of length m is written $\boldsymbol{\jmath_m}$. We call two linear codes **isomorphic** if they can be obtained from one another by permuting the coordinate positions. An **automorphism** of a code C is an isomorphism from C to C. The automorphism group will be denoted by $\mathrm{Aut}(C)$. An **information set** for C is the set of k coordinate positions of a set of k linearly independent columns of a generator matrix for C. The remaining coordinates are called a **check set**.

For any finite field $\mathbb{F}_q$ of order q, the set of points and r-dimensional subspaces of an m-dimensional projective geometry forms a 2-design which we will denote by $\boldsymbol{PG_{m,r}(\mathbb{F}_q)}$. The **automorphism group** of each of these designs is the full projective semi-linear group, $P\Gamma L_{m+1}(\mathbb{F}_q)$ and is 2-transitive on points. The codes of these designs are subfield subcodes of the generalized Reed-Muller codes: see [1, Chapter 5] for a full treatment.

2.2 Graphs and codes

The **graphs**, $\Gamma = (V, E)$ with vertex set V and edge set E, are simple. If $X, Y \in V$ and X and Y are adjacent, we write $X \sim Y$, and XY or $[X, Y]$ for the **edge** in E that they define. The **set of neighbours** of $X \in V$ is denoted by $\mathcal{N}(X)$, and the **valency of** X is $|\mathcal{N}(X)|$. Γ is **regular** if all the vertices have the same valency. A **path** of length r from vertex X to vertex Y is a sequence X_i, for $0 \le i \le r-1$, of distinct vertices with $X = X_0$, $Y = X_{r-1}$, and $X_{i-1} \sim X_i$ for $1 \le i \le r-1$. It is **closed** of length r if $X \sim Y$, in which case we write it $(X_0, \ldots, X_{r-1})$. The graph is **connected** if there is a path between any two vertices. A **perfect matching** is a set S of disjoint edges such that every vertex is on exactly one member of S. An **adjacency matrix** A is a $|V| \times |V|$ matrix with entries a_{ij} such that $a_{ij} = 1$ if vertices X_i and X_j are adjacent, and $a_{ij} = 0$ otherwise. An **incidence matrix** is a $|V| \times |E|$ matrix B with $b_{ij} = 1$ if the vertex labelled by i is on the edge labelled by j, and $b_{ij} = 0$ otherwise. If Γ is regular with valency k, then the 1-$(|E|, k, 2)$ design with incidence matrix B is called the **incidence design** of Γ. The **neighbourhood design** of Γ is the symmetric 1-$(|V|, k, k)$ design formed by taking the points to be the vertices of the graph and the blocks to be the sets of neighbours of a vertex, for each vertex, i.e. an adjacency matrix as an incidence matrix for the design. The **line graph** of Γ is the graph $L(\Gamma)$ with E as vertex set and where adjacency is defined so that e and f in E, as vertices, are adjacent in $L(\Gamma)$ if e and f as edges of Γ share a vertex in Γ. A **strongly regular graph** Γ of type (n, k, λ, μ) is a regular graph on $n = |V|$ vertices, with valency k which is such that any two adjacent vertices are together adjacent to λ vertices and any two non-adjacent vertices are together adjacent to μ vertices.

The **code** of Γ over a finite field F is the row span of an adjacency matrix A over the field F, denoted by $C_F(\Gamma)$ or $C_F(A)$. The dimension of the code is the rank of the matrix over F, also written $\mathrm{rank}_p(A)$ if $F = \mathbb{F}_p$, in which case we will speak of the **p-rank** of A or Γ, and write $C_p(\Gamma)$ or $C_p(A)$ for the code. It is also the code over F_p of the neighbourhood design. Similarly, if G is an incidence matrix for Γ, $C_p(G)$ denotes the row span of G over F_p. If L is an adjacency matrix for $L(\Gamma)$ where Γ is regular, then

$$G^T G = L + 2I_{|E|}. \tag{1}$$

We need some of the notions of edge connectivity. If $\Gamma = (V, E)$ is a connected graph, then an **edge-cut** of Γ is a subset $S \subseteq E$ such that removing the edges in S renders the new graph $\Gamma - S$ disconnected. The **edge-connectivity** of Γ, denoted by $\lambda(\Gamma)$, is the minimum cardinality of an edge-cut of Γ. If Γ is k-regular then $\lambda(\Gamma) \le k$; Γ is **super-λ** if $\lambda(\Gamma) = k$ and every minimal edge-cut consists of the edges incident with some vertex. If $\Gamma - S$ has only nontrivial components, i.e., components with at least two vertices, then S is a **restricted edge-cut**. The minimal cardinality of a restricted edge-cut is the **restricted edge-connectivity**, denoted by $\lambda'(\Gamma)$. If Γ is k-regular then $\lambda'(\Gamma) \le 2k-2$. Γ is **super-λ'** if $\lambda'(\Gamma) = 2k-2$ and every minimal restricted edge-cut consists of the edges incident with some edge. A **restricted bipartition set** is a bipartition set $S \subseteq E$ such that $\Gamma - S$ has a nontrivial bipartite component, i.e., a component that is bipartite and contains more than one vertex. The minimum cardinality of a restricted bipartition set of Γ is denoted by $\lambda'_{bip}(\Gamma)$. If Γ is k-regular then $\lambda'_{bip}(\Gamma) \le 2k-2$. Γ is **super-λ'_{bip}** if $\lambda'_{bip}(\Gamma) = 2k-2$ and every minimum restricted bipartition set consists of the edges incident with some edge. The minimum number of edges of Γ whose removal renders the graph bipartite is denoted by $b(\Gamma)$.

If $\Gamma = (V, E)$ is a graph, X a vertex, $\mathcal{N}(X)$ its neighbours, then we write

$$\overline{X} = \{[X, Y] \mid Y \in V, Y \sim X\} = \{[X, Y] \mid Y \in \mathcal{N}(X)\}, \tag{2}$$

i.e. the edges that correspond to non-zero entries in the row labelled by X of an incidence matrix G for Γ. Thus $C_p(G)$ is generated by the vectors $v^{\overline{X}} = \sum_{Y \in \mathcal{N}(X)} v^{[X,Y]}$.

For the line graph $L(\Gamma)$ we have $\mathcal{N}([X, Y]) = \{[X, Z] \mid Z \neq Y\} \cup \{[Y, Z] \mid Z \neq X\}$, for the neighbours of $[X, Y]$.

Lemma 1. *Let Γ be a graph, $L(\Gamma)$ its line graph, and G an incidence matrix for Γ. If $\pi = (X_1, \ldots, X_l)$ is a closed path in Γ, then $w(\pi) = \sum_{i=1}^{l-1} v^{[X_i, X_{i+1}]} + v^{[X_l, X_1]} \in C_2(G)^{\perp}$.*

Proof: The proof is clear. □

We will need the following result from [8]:

Result 1. *Let $\Gamma = (V, E)$ be a regular graph with valency k and $\mathcal{G}$ the 1-$(|E|, k, 2)$ incidence design for Γ. Then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\mathcal{G})$.*

We need also the following from [23, Result 2]:

Result 2. *Let $\Gamma = (V, E)$ be a graph, G an incidence matrix for Γ, $C_p(G)$ the row-span of G over $\mathbb{F}_p$. If Γ is connected then $\dim(C_2(G)) = |V| - 1$, and if Γ is connected and has a closed path of odd length ≥ 3, then $\dim(C_p(G)) = |V|$ for odd p.*

2.3 Permutation decoding

Permutation decoding was first developed by MacWilliams [27] and involves finding a set of automorphisms of a code called a PD-set. The method is described in MacWilliams and Sloane [28, Chapter 16, p. 513] and Huffman [17, Section 8]. In [18] and [24] the definition of PD-sets was extended to that of s-PD-sets for s-error-correction:

Definition 1. *If C is a t-error-correcting code with information set $\mathcal{I}$ and check set $\mathcal{C}$, then a* **PD-set** *for C is a set $\mathcal{S}$ of automorphisms of C which is such that every t-set of coordinate positions is moved by at least one member of $\mathcal{S}$ into the check positions $\mathcal{C}$.*

For $s \leq t$ an **s-PD-set** *is a set $\mathcal{S}$ of automorphisms of C which is such that every s-set of coordinate positions is moved by at least one member of $\mathcal{S}$ into $\mathcal{C}$.*

The algorithm for permutation decoding is given in [17]. There is a bound on the minimum size that the set S may have, due to Gordon [16], from a formula due to Schönheim [29], and quoted and proved in [17]. The formula can be adapted to s-PD-sets for $s \leq t$ by replacing t by s in the formula: see, for example, [11]

3 Incidence matrices of odd graphs

The odd graphs $\mathcal{O}_k$ for $k \geq 2$ are the uniform subset graphs $G(2k+1, k, 0)$, i.e. if Ω is a set of size $2k+1$, the vertex set of $\mathcal{O}_k$ is the set $\Omega^{\{k\}}$ of subsets of size k of Ω, with two vertices being adjacent if the two k-subsets intersect in the empty set. The graphs are regular, of valency $\nu = k+1$, and $\mathrm{Aut}(\mathcal{O}_k) = S_{2k+1}$, acting transitively on vertices and on edges. The set of edges of $\mathcal{O}_k$ is denoted by $\mathcal{P}_k$ and is the point set of the 1-$(\frac{k+1}{2}\binom{2k+1}{k}, k+1, 2)$ incidence design $\mathcal{G}_k$ of $\mathcal{O}_k$. Thus $\mathcal{O}_k = (\Omega^{\{k\}}, \mathcal{P}_k)$.

General theorems in [6, 5], along with special results for $\mathcal{O}_k$, can be used to prove the following proposition; these depend on results from the literature on edge cuts.

Proposition 1. *For $k \geq 2$, let G_k be a $\binom{2k+1}{k} \times \frac{k+1}{2}\binom{2k+1}{k}$ incidence matrix for the odd graph $\mathcal{O}_k$. For p any prime, let $\varepsilon_p = 0$ if p is odd, $\varepsilon_2 = 1$.*

Then for any prime p, $C_p(G_k)$ is a $\left[\frac{k+1}{2}\binom{2k+1}{k}, \binom{2k+1}{k} - \varepsilon_p, k+1\right]_p$ code.

If $k \geq 3$, the minimum words are the scalar multiples of the rows of G_k, there are no words of weight i where $k+1 < i < 2k$, and the words of weight $2k$ are the scalar multiples of the differences of two rows corresponding to two adjacent vertices.

If $p = 2$, the same is true for $k = 2$. For p odd, $C_p(G_2)$ has more words of weight 3.

We need a lemma to employ the results of [6, 5] before we can prove this proposition. First note that for $k \geq 2$, since $\mathcal{O}_k$ is both vertex and edge transitive, has no triangles and is not bipartite, from Result 9 in [5], $\mathcal{O}_k$ is both super-λ and super-λ'. We also need to show that, for $k \geq 3$, $\mathcal{O}_k$ is super-λ'_{bip}, and this will hold if it is super-λ' and if $b(\mathcal{O}_k) > 2\nu - 2 = 2k$. Thus we need only check the bound on $b(\mathcal{O}_k)$. The necessary terminology, including $b(\Gamma)$, that we use here for edge-connectivity is defined in Section 2.2.

Lemma 2. *Let $\Gamma = (V, E)$ be a ν-regular connected graph which is such that every edge is on x m-cycles, where $x > 0$ and $m \geq 3$ is odd. Then $b(\Gamma) \geq \frac{1}{2m}|V|\nu$.*

$\mathcal{O}_k$ is connected for $k \geq 2$, and if $k \geq 3$, $b(\mathcal{O}_k) > 2k$.

Proof: The number of m-cycles is $t = \frac{1}{2m}|V|\nu x$. A set S of s edges removed from E such that $\Gamma - S$ is bipartite must destroy all the m-cycles; since each edge is on x m-cycles, at most sx m-cycles are destroyed, so $sx \geq t$. In particular $b(\Gamma) \geq \frac{1}{2m}|V|\nu$.

Now let $\Gamma = \mathcal{O}_k$. Then Γ has cycles of length $m = 2k+1$: if $\sigma = (1, 2, \ldots, 2k+1)^{-1} \in \mathrm{Aut}(\Gamma)$, $X_0 = \{1, 2, \ldots, k\}$, $Y_0 = \{k+1, k+2, \ldots, 2k\}$, and $X_i = X_0^{\sigma^i}$, $Y_i = Y_0^{\sigma^i}$, for $1 \leq i \leq k$, then $Y_k = X_0$, and

$$(X_0, Y_0, X_1, Y_1, \ldots, X_{k-1}, Y_{k-1}, X_k)$$

is a closed path of length $2k+1$ in Γ. To show that Γ is connected, note first that $\mid X_0 \cap X_i \mid = k - i$ for $0 \leq i \leq k$, so to show that we can find a path between any two vertices, we need only show that X_0 is connected to any vertex Z, where $\mid X_0 \cap Z \mid = r$, and $0 \leq r \leq k-1$. Clearly in S_{2k+1} we can map the pair X_0, X_{k-r} to X_0, Z, which proves the assertion.

From the first part of the lemma, with $m = 2k+1$, $|V| = \binom{2k+1}{k}$, and $\nu = k+1$, we have $b(\mathcal{O}_k) \geq \frac{1}{2(2k+1)}\binom{2k+1}{k}(k+1)$. This is easily shown to be strictly greater than $2k$ for $k \geq 3$. □

Note that the inequality in the lemma is not true for $k = 2$ and in fact the codes $C_p(\mathcal{O}_2)$ for p odd do not satisfy these properties, as noted before.

The proof of the proposition now follows from the facts that $\mathcal{O}_k$ is connected, super-λ and super-λ' for $k \geq 2$, and super-λ'_{bip} for $k \geq 3$, and Theorems 11, 18 of [5].

4 Binary codes of line graphs of odd graphs

Let G_k denote an incidence matrix for $\mathcal{O}_k$, and L_k an adjacency matrix for the line graph $L(\mathcal{O}_k)$. For binary codes, we have

$$(G_k)^T G_k = L_k.$$

The following result is deduced from results in [22, 7, 8]:

Result 3. *Let $\Gamma = (V, E)$ be a connected graph, G a $|V| \times |E|$ incidence matrix for Γ, and L an adjacency matrix for $L(\Gamma)$. Let $E(G)$ denote the binary code spanned by the differences of all pairs of rows of G. Then $C_2(L) = E(G)$ and $C_2(L) = C_2(G)$ if and only if $\mathbf{j}_{|V|} \notin C_2(G^T)$. Further, if Γ has a perfect matching then $\mathbf{j}_{|V|} \in C_2(G^T)$.*

The result applies to $\mathcal{O}_k$ so we need to determine when $\mathbf{j}_{\binom{2k+1}{k}} \in C_2(G_k^T)$. By [26] every vertex-transitive graph on an even number of vertices has a perfect matching, so that this will be true for $\mathcal{O}_k$, $k \geq 2$. Thus we need only establish when the number of vertices is even.

Proposition 2. *For $k \geq 2$, let $G_k = [g_{i,j}]$ be a $\binom{2k+1}{k} \times \frac{k+1}{2}\binom{2k+1}{k}$ incidence matrix for $\mathcal{O}_k$, $E(G_k) = \langle g_{i,j} - g_{i,m} \mid 1 \leq i \leq 2k+1 \rangle$ over $\mathbb{F}_2$, and L_k an adjacency matrix for $L(\mathcal{O}_k)$. Then $E(G_k) = C_2(L_k)$ and if $k = 2^l - 1$ for some $l \geq 2$, then $C_2(L_k) = C_2(G_k)$; otherwise $C_2(L_k)$ has codimension 1 in $C_2(G_k)$*

and is a $\left[\frac{k+1}{2}\binom{2k+1}{k}, \binom{2k+1}{k} - 2, 2k\right]_2$ *code and the words of weight* $2k$ *are the differences of two rows of* G_k *corresponding to adjacent vertices.*

Proof: If $k = 2^l - 1$ for some $l \geq 2$ then by [10, page 78], $\binom{2k+1}{k}$ is odd, so $\mathbf{J}_{\binom{2k+1}{k}} \notin C_2(G_k^T)$ since this is an even-weight code. If k is even, the sum of all the rows of G_k^T is $\mathbf{J}_{\binom{2k+1}{k}}$; if k is odd, $k \neq 2^l - 1$ for any $l \geq 2$, then by [10, page 78], $\binom{2k+1}{k}$ is even. The statement about the minimum weight and the nature of the minimum words follows from Proposition 1. This completes the proof. □

Note: The fact that $\dim(C_2(L_k))$ is even (see for example [15, Proposition 2.1]) could be used in the proof. Also, if $C_2(L_k) \neq C_2(G_k)$ then $C_2(L_k)$ cannot contain any vectors that are a sum of an odd number of rows of G_k.

5 Binary hulls for $C_2(G_k)$

Recall that $\mathrm{Hull}(C) = C \cap C^\perp$ for C any code. Thus it is a self-orthogonal code. A study of the hull of a code or design often leads to some defining characters of the structure: see [1, 13] for more about hulls. Hulls of codes from incidence matrices of regular connected graphs were studied in [12]. For the binary hull of an adjacency matrix of $L(\Gamma)$ we have the following:

Lemma 3. *Let* Γ *be a graph,* G *an incidence matrix for* Γ*,* $C = C_2(G)$*,* $H = \mathrm{Hull}(C)$*,* L *an adjacency matrix for* $L(\Gamma)$*,* $CL = C_2(L)$*, and* $HL = \mathrm{Hull}(CL)$*. Then either* H *and* HL *are equal, or one has codimension* 1 *in the other.*

Proof: By Result 3, $C_2(L) = E(G)$, spanned by the differences of the rows of G. Suppose $H \neq HL$. If there is $w \in H$ but $w \notin HL$, then $w = \sum_{X \in \mathcal{X}} v^{\overline{X}}$ where $|\mathcal{X}|$ is odd, and $w \in C^\perp$ implies that $(w, v^{\overline{X}}) = 0$ for all vertices X. We show that $HL \subset H$ and that $H = \langle w, HL \rangle$. If $u \in HL$ then $(u, v^{\overline{X}}) = c$, a constant for all vertices X. Now $(u, w) = 0$ since $w \in C^\perp$, so $(u, w) = (u, v^{\overline{X}} + v)$ where $X \in \mathcal{X}$ and $v \in CL$, so that $0 = (u, v^{\overline{X}}) = c$, and thus $u \in C^\perp$ and thus in H, so $HL \subset H$. For any $h \in H$ for which $h \notin HL$, $h \neq w$, h is a sum of an odd number of rows of G, so $h + w \in CL$ and $h + w \in C^\perp \subseteq CL^\perp$, and hence $h + w \in HL$.

If $w \in HL$ but $w \notin H$, then $w \notin C^\perp$ and $(w, v^{\overline{X}}) = 1$ for all vertices X. If $u \in H$ then $(u, v^{\overline{X}}) = 0$ for all vertices X. If $u = \sum_{X \in \mathcal{S}} v^{\overline{X}}$, then $0 = (u, w) = \sum_{X \in \mathcal{S}} (w, v^{\overline{X}}) = |\mathcal{S}|$, which is thus even, so $u \in CL$ and thus in HL. So $H \subset HL$. To show that $HL = \langle w, H \rangle$: if $v \in HL, v \notin H$ then $(v, v^{\overline{X}}) = 1$ for all X, so $(v + w, v^{\overline{X}}) = 0$, so $v + w \in H$. □

We show below that the binary hulls from the incidence matrices of the graphs $\mathcal{O}_k$, and those from the adjacency matrices of their line graphs, are non-trivial for all k; this contrasts with the hull from the adjacency matrix of $\mathcal{O}_k$, as studied in [10], which is always trivial. The p-ary hulls for p odd do not have such a uniform characterization and the properties vary according to p and the parameters of the graph (see [12]).

Using the formulae $\mathrm{rank}_2(A) = \binom{2k}{k}$ and $\mathrm{rank}_2(A + I) = \binom{2k}{k-1} + \binom{2k-1}{k-1} - 2^{k-1}$ for A an adjacency matrix for $\mathcal{O}_k$, the following was proved in [12, Result 7]:

Result 4. *For* $k \geq 2$*, let* G_k *be an incidence matrix for* $\mathcal{O}_k$*,* $H_k = \mathrm{Hull}(C_2(G_k))$*. Then*

$$\dim(H_k) = \begin{cases} \binom{2k-1}{k} + 2^{k-1} - 1 & \text{for } k \text{ even} \\ \binom{2k}{k-1} - 1 & \text{for } k \text{ odd.} \end{cases}$$

Note: For $HL_k = \mathrm{Hull}(C_2(L_k))$, if $k = 2^l - 1$ where $l \geq 2$ then $H_k = HL_k$, by Proposition 2. For other k we have equality or $\dim(HL_k) = \dim(H_k) \pm 1$, by Lemma 3.

Proposition 3. *For* $k \geq 2$*, let* G_k *be an incidence matrix for* $\mathcal{O}_k$ *with vertex set* $\Omega^{\{k\}}$*, where* $\Omega = \{1, 2, \ldots, 2k+1\}$*,* $H_k = \mathrm{Hull}_2(C_2(G_k))$*,* L_k *an adjacency matrix for* $L(\mathcal{O}_k)$*, and* $HL_k = \mathrm{Hull}(C_2(L_k))$*. Then the minimum weight of* H_k *and* HL_k *is at least* $2k + 2$*.*

If k is even, let $S = \{\{2i-1, 2i\} \mid 1 \le i \le k\}$, and $\mathcal{X} = \{X \mid X \in \Omega^{\{k\}}, |X \cap s| = 1, \forall s \in S\}$. Then $w = \sum_{X\in\mathcal{X}} v^{\overline{X}} \in H_k, HL_k$, $\mathrm{wt}(w) = k2^k$, and there are $\prod_{i=1}^{k} \binom{2i+1}{2}/k!$ such words.

Proof: (1). H_k is a self-orthogonal binary code, so its words all have even weight. Since it is not $C_2(G_k)$, it must have minimum weight at least $2k$. The words of weight $2k$ are the vectors $v^{\overline{X}} - v^{\overline{Y}}$ where $X \sim Y$. If $w = v^{\overline{X}} - v^{\overline{Y}} \in H_k$ then $(w, v^{\overline{X}}) = k+1+1 = 0$, so k must be even. If $X \sim Z$ and $Z \neq Y$ then $Z \not\sim Y$ (since there are no triangles) and hence $(w, v^{\overline{Z}}) = 1$, which is a contradiction. So H_k has minimum weight at least $2k+2$.

Similarly, HL_k has minimum weight at least $2k$. If $w = v^{\overline{X}} - v^{\overline{Y}} \in HL_k$ then $(w, v^{\overline{Z}} - v^{\overline{W}}) = 0$ for all Z, W, $Z \sim W$. If $Z \sim X$ then $X \not\sim W$, $Z \not\sim Y$ and $W \not\sim Y$ since there are no paths of length 3 or 4 in $\mathcal{O}_k$. So $(w, v^{\overline{Z}} - v^{\overline{W}}) = (v^{\overline{X}} - v^{\overline{Y}}, v^{\overline{Z}} - v^{\overline{W}}) = 1$ so $w \notin HL_k$. So HL_k has minimum weight at least $2k+2$.

Suppose $k \ge 2$ is even. Let $\Omega^* = \Omega \setminus \{2k+1\}$, and for $X \in \mathcal{X}$, let X^c denote its complement in Ω^*. Then $X^c \in \mathcal{X}$ and for $X, Y \in \mathcal{X}$, $X \not\sim Y$ unless $Y = X^c$. It follows that $\mathrm{wt}(w) = k2^k$. It also follows that, for $X \in \mathcal{X}$, $(w, v^{\overline{X}}) = (v^{\overline{X}}, v^{\overline{X}}) + (v^{\overline{X^c}}, v^{\overline{X}}) = k+1+1 \equiv 0$.

If $X \notin \mathcal{X}$ and $X \subset \{1, \ldots, 2k\}$, then $X \supseteq s$ for some $s \in S$. Since every $Y \in \mathcal{X}$ meets every $s \in S$, we have $X \not\sim Y$ for $Y \in \mathcal{X}$, and so $(w, v^{\overline{X}}) = 0$. If $2k+1 \in X$, then X cannot meet all the $s \in S$. If $X \supseteq s$ for some $s \in S$ then $(w, v^{\overline{X}}) = 0$ as in the previous case. Suppose X does not contain any $s \in S$; then X must meet $k-1$ of the $s \in S$ exactly once, and be disjoint from exactly one, $\{2j-1, 2j\}$. If $X = \{a_1, \ldots, a_{k-1}, 2k+1\}$, where $a_i \in s_{r_i} \in S$ then $X \sim \{s_{r_i} \setminus \{a_i\} \mid 1 \le i \le k-1\} \cup \{2j-1\}$ and $X \sim \{s_{r_i} \setminus \{a_i\} \mid 1 \le i \le k-1\} \cup \{2j\}$, and $X \not\sim Y$ for the other members of $\mathcal{X}$. Thus $(w, v^{\overline{X}}) = 0$ and so $w \in H_k$. Then also $w \in HL_k$ since it is a sum of an even number of rows of G_k. That the number of such vectors is as stated follows from a simple count. □

6 Petersen graph, $PG_3(\mathbb{F}_2)$ and the Fano plane

$\mathcal{O}_2$ is the strongly regular Petersen graph with parameters $(10, 3, 0, 1)$. We show how the points of the projective geometry $PG_3(\mathbb{F}_2)$ can be put into correspondence with the 15 edges of $\mathcal{O}_2$ such that the vertices correspond to pencils of lines, and such that $\mathrm{Hull}_2(C_2(G_2))^\perp$ is the code of the 2-$(15, 3, 1)$ Steiner triple system defined by the points and lines of $PG_3(\mathbb{F}_2)$. Again, G_2 is an incidence matrix for $\mathcal{O}_2$, and in this section we write $\Gamma = \mathcal{O}_2$.

From Magma [2, 4] we found that $H_2 = \mathrm{Hull}_2(C_2(G_2))$ is a $[15, 4, 8]_2$ code, with weight distribution $(< 0, 1 >, < 8, 15 >)$, while $H_2^\perp$ is a $[15, 11, 3]_2$ code, and has weight distribution

$$< 0, 1 >, < 3, 35 >, < 4, 105 >, < 5, 168 >, < 6, 280 >, < 7, 435 >,$$

together with the complements since clearly $\jmath_{15} \in H_2^\perp$. We show how these parameters can be explained. In Proposition 3, 15 words of weight 8 in H_2 are described. Define notation for these words as follows: let $S = \{a, b, c, d\}$ be a 4-subset of $\Omega = \{a, b, c, d, e\}$. For each of the three partitions p_i of S into two disjoint subsets of size 2, we define a word of the hull H_2 as follows: if $p_i = \{\{a, b\}, \{c, d\}\}$, and $P = \{a, b\}$, $Q = \{c, d\}$, then

$$v(P, Q) = v(Q, P) = v(p_i) = v(a, b; c, d) = v(c, d; a, b) = v^{\overline{\{a,c\}}} + v^{\overline{\{a,d\}}} + v^{\overline{\{b,c\}}} + v^{\overline{\{b,d\}}}, \tag{3}$$

has support

$$\{[\{a, c\}, \{e, b\}], [\{a, d\}, \{e, b\}], [\{a, d\}, \{e, c\}], [\{b, d\}, \{e, c\}],$$
$$[\{b, d\}, \{e, a\}], [\{b, c\}, \{e, a\}], [\{b, c\}, \{e, d\}], [\{a, c\}, \{e, d\}]\},$$

and is $w(\pi)$ of Lemma 1 for the 8-cycle $\pi = (\{a, c\}, \{e, b\}, \{a, d\}, \{e, c\}, \{b, d\}, \{e, a\}, \{b, c\}, \{e, d\})$.

Lemma 4. *The supports of the* 15 *words of weight* 8 *from Equation* (3) *in H_2 form a* 2-$(15, 8, 4)$ *symmetric design $\mathcal{D}$ whose complement is a* 2-$(15, 7, 3)$ *symmetric design $\mathcal{D}^c$. Further, $C_2(\mathcal{D})$ is a $[15, 4, 8]_2$ code inside H_2, and $C_2(\mathcal{D}^c)$ is a $[15, 5, 7]_2$ code.*

Proof: Consider the 15 words of weight 8 from Equation (3). We write down the eight blocks containing a point $X = [P, Q]$ where $P = \{a, b\}$ and $Q = \{c, d\}$. Then X will be in the support of words from the 4-sets $\Omega \setminus \{x\}$ for each $x \in \{a, b, c, d\}$, two from each of these 4-sets as follows:

$$v(e, c; b, d), v(e, d; b, c); v(e, c; a, d), v(e, d; a, c); v(e, a; b, d), v(e, b; a, d); v(e, a; b, c), v(e, b; a, c). \quad (4)$$

We show now that any other point $Y = [R, S]$ is together on four blocks with X. There are three cases: one if $R \cup S = P \cup Q$, and two if $|(R \cup S) \cap (P \cup Q)| = 3$. In the first case, we can assume $Y = [\{a, c\}, \{b, d\}]$. Then from the words in Equation (4), Y is in:

$$v(e, d; b, c), v(e, c; a, d), v(e, b; a, d), v(v(e, a; b, c).$$

Now suppose $Y = [\{a, b\}, \{c, e\}]$. Then from the words in Equation (4), Y is in

$$v(e, d; b, c); v(e, d; a, c); v(e, a; b, d), v(e, b; a, d).$$

If $Y = [\{a, c\}, \{b, e\}]$, then from the words in Equation (4), Y is in

$$v(e, c; b, d), v(e, d; b, c); v(e, c; a, d); v(e, a; b, d).$$

This completes all the possibilities. Thus we have a 2-$(15, 8, 4)$ symmetric design $\mathcal{D}$, with complement $\mathcal{D}^c$ a 2-$(15, 7, 3)$ symmetric design.

To prove the result concerning the codes, we first show explicitly that the sum of any two weight-8 vectors from blocks of $\mathcal{D}$ is another block. This is done directly using Equation (3) for the various cases. Firstly note that for any two partitions of a set S of size 4, the binary sum of two incidence vectors $v(p_i)$ and $v(p_j)$ is the third $v(p_k)$. Next, if all the elements of Ω appear, first we have the case $v(P, Q)$ and $v(P, R)$, with $P = \{a, b\}$, $Q = \{c, d\}$, $R = \{c, e\}$, i.e.

$$v(P, Q) + v(P, R) = v(a, b; c, d) + v(a, b; c, e) = v(a, b; d, e).$$

Otherwise if we have four vertices, i.e $[P, Q]$ and $[R, S]$, where P, Q, R, S are all distinct, then there is essentially only one case, i.e. with $[P, Q] = [\{a, b\}, \{c, d\}]$ and $[R, S] = [\{a, c\}, \{d, e\}]$. Then

$$v(a, b; c, d) + v(a, c; d, e) = v(a, e; b, d).$$

Thus $C_2(\mathcal{D})$ consists of 15 words of weight 8, together with the zero vector, so it is $[15, 4, 8]_2$ code as asserted. To get $C_2(\mathcal{D}^c)$, add $\boldsymbol{\jmath}_{15}$ and get a $[15, 5, 7]_2$ code with weight distribution

$$< 0, 1 >, < 7, 15 >, < 8, 15 >, < 15, 1 >,$$

which completes the proof. □

Note: 1. If $\mathcal{H}_r$ denotes the binary Hamming code of length $2^r - 1$, dimension $2^r - 1 - r$ (see, for example, [1, Section 2.5]), then $C_2(\mathcal{D}) = \mathcal{H}_4^\perp$, the binary simplex code.
2. The 15 weight-8 words in H_2 correspond to the 15 8-cycles in $\mathcal{O}_2$ and the 15 weight-7 words in $\langle \boldsymbol{\jmath}_{15}, H_2 \rangle$ to the complements of these in the set of edges of $\mathcal{O}_2$.

Proposition 4. *Let G_2 denote a 10×15 incidence matrix for $\mathcal{O}_2$, $C = C_2(G_2)$, $H_2 = \mathrm{Hull}(C)$. Then $C + C^\perp = H_2^\perp$ contains 35 vectors of weight 3 and the set of supports form the blocks of a 2-$(15, 3, 1)$ Steiner triple system $\mathcal{S}$ that is the design $PG_{3,1}(\mathbb{F}_2)$ of points and lines in the projective geometry $PG_3(\mathbb{F}_2)$. Furthermore, $C_2(\mathcal{S}) = C + C^\perp = H_2^\perp = \mathcal{H}_4$.*

Proof: From Proposition 1, C is a $[15, 9, 3]_2$ code. Let $\Omega = \{a, b, c, d, e\}$. We show how to construct 35 weight-3 vectors in $C + C^\perp$. There are ten from the rows of C. We will call these **words of type I**, corresponding to the ten vertices of $\Gamma = \mathcal{O}_2$.

Next, note that $\pi = (\{a, b\}, \{c, d\}, \{e, b\}, \{a, d\}, \{c, b\}, \{e, d\})$ is a 6-cycle in Γ and $w(\pi) \in C^\perp$ by Lemma 1. Then

$$w(\pi) + v^{\overline{\{c,d\}}} + v^{\overline{\{a,d\}}} + v^{\overline{\{d,e\}}} = v^{[\{c,d\},\{a,e\}]} + v^{[\{a,d\},\{c,e\}]} + v^{[\{d,e\},\{a,c\}]} \in C + C^\perp.$$

Thus we get five weight-3 vectors of this form by taking the three partitions into pairs of each of the five 4-subsets. We denote the resultant weight-3 vector by $t(x)$ if x is the omitted element. Thus

$$t(b) = v^{[\{c,d\},\{a,e\}]} + v^{[\{a,d\},\{c,e\}]} + v^{[\{d,e\},\{a,c\}]}. \tag{5}$$

We call these **words of type II**, and they correspond to the five "spokes" of the graph when drawn in its usual representation (see Figure 1).

The remaining 20 weight-3 vectors are obtained in a similar way from the words $w(\pi)$ of weight 5 in $C^\perp$ from closed paths π of length 5 as described in the proof of Lemma 2. Thus for

$$\pi = (\{a,b\},\{c,d\},\{a,e\},\{b,c\},\{d,e\}),$$

$$w(\pi) + v^{\overline{\{c,d\}}} + v^{\overline{\{b,c\}}} = v^{[\{d,c\},\{b,e\}]} + v^{[\{d,a\},\{b,c\}]} + v^{[\{d,e\},\{b,a\}]} = f_1(d,b) \in C + C^\perp. \tag{6}$$

Then

$$f_1(d,b)^{(a,e)} = f_2(d,b) = v^{[\{d,c\},\{b,a\}]} + v^{[\{d,e\},\{b,c\}]} + v^{[\{d,a\},\{b,e\}]} \in C + C^\perp, \tag{7}$$

and it is clear that for every pair x, y of elements from Ω we get exactly two weight-3 vectors of this type, $f_i(x,y)$, $i = 1,2$, where each point of the support has the form $[P,Q]$ where $x \in P, y \in Q$. This gives another $\binom{5}{2} \times 2 = 20$ weight-3 vectors. We call the words $f_i(a,b)$ **words of type III**. Note that $\mathrm{Supp}(f_1(d,b)) \cup \mathrm{Supp}(f_2(d,b))$ is the 6-cycle $(\{a,b\},\{d,c\},\{b,e\},\{a,d\},\{b,c\},\{d,e\})$, and any 6-cycle in the graph will give two lines in the geometry from the two sets of three alternate edges.

We now show that the set of supports of these weight-3 vectors form the blocks of a 2-$(15,3,1)$ design. First notice that the replication number r is 7, since $[\{a,b\},\{c,d\}]$ is in the support of

$$v^{\overline{\{a,b\}}}, v^{\overline{\{c,d\}}}, t(e), f_i(a,c), f_j(a,d), f_k(b,c), f_l(b,d),$$

where i,j,k,l are 1 or 2. There are two of type I, one of type II, and four of type III, the latter corresponding to alternate edges of each of the four 6-cycles that contain $[\{a,b\},\{c,d\}]$.

Now take two points, $X = [P,Q], Y = [R,S]$. If $P = R$, then clearly $\overline{P}$ is the only block containing X, Y. If $|P \cup Q \cup R \cup S| = 4$ and if z is the element that does not appear, then $t(z)$ is the only block containing both X and Y. If $P \cup Q \cup R \cup S = \Omega$, then suppose $X = [\{a,b\},\{c,d\}]$. Then Y has the form $[\{e,a\},\{c,b\}]$ since it is not of a type previously looked at. Thus X and Y are together in $f_i(a,c)$ for $i = 1$ or 2, again giving a unique block.

Thus $\mathcal{S}$ is a 2-$(15,3,1)$ design. To show $\mathcal{S} = PG_{3,1}(\mathbb{F}_2)$, we look at $C_2(\mathcal{S})$. Clearly $C_2(\mathcal{S}) \subseteq C + C^\perp = H_2^\perp$, since it is spanned by vectors from $H_2^\perp$. From Lemma 4, $\dim(H_2) \geq 4$, so $\dim(H_2^\perp) \leq 11$, and thus $\dim(C_2(\mathcal{S})) \leq 11$. However, by a theorem of Doyen, Hubaut and Vandensavel (see [1, Theorem 8.2.1, page 297]), the binary code of a Steiner triple system with these parameters has dimension at least 11, and equal to 11 only if it is $PG_{3,1}(\mathbb{F}_2)$. Thus $\mathcal{S} = PG_{3,1}(\mathbb{F}_2)$ and $C_2(\mathcal{S}) = H_2^\perp$. □

Note: A line in $PG_3(\mathbb{F}_2)$ that is the support of a word of type X will be called a **line of type X**, where X is I, II, or III.

Corollary 1. *The design $\mathcal{D}^c$ of Lemma 4 is $PG_{3,2}(\mathbb{F}_2)$, the design of points and planes in the projective geometry $PG_3(\mathbb{F}_2)$. Further,* $\mathrm{Aut}(\mathcal{D}^c) = \mathrm{Aut}(\mathcal{D}) = \mathrm{Aut}(\mathcal{S}) = PGL_4(\mathbb{F}_2) \cong A_8$.

Proof: From [3, Proposition 2] we know that the minimum words of $C_2(\mathcal{S})^\perp$ are the incidence vectors of the complements of the Fano planes in $PG_3(\mathbb{F}_2)$. These are precisely the words of weight-8 in $H_2 = CS^\perp$. Thus the 15 words of weight 8 (one from each 8-cycle) are the complements of the 15 Fano planes, and so $\mathcal{D}^c = PG_{3,2}(\mathbb{F}_2)$. The automorphism group follows. □

Note: It can be shown that if L_2 is an adjacency matrix for $L(\mathcal{O}_2)$, then $\mathrm{Hull}(C_2(L_2)) = \mathrm{Hull}(C_2(G_2))$.

We now fit the points of $PG_3(\mathbb{F}_2)$ onto the edges of the Petersen graph $\Gamma = \mathcal{O}_2$, with vertices labelled $\{i,j\}$ for $i,j \in \{1,\ldots,5\}$, $i \neq j$, as shown in Figure 1. We will first place a Fano plane $\mathcal{F}$ with point set $A,\ldots,G$, and lines as shown in Fig. 1. The remaining points of $PG_3(\mathbb{F}_2)$ will be labelled $H, I, \ldots, O$ and we will fit these onto the edges of Γ after we have placed $\mathcal{F}$. Since, from the proof of Proposition 4, any point of $PG_3(\mathbb{F}_2)$ is on two lines of type I, one of type II, and four of type III, so dually each Fano plane

contains two lines of type I, one of type II, and four of type III. Recall that the Fano planes correspond to the complements of the 8-cycles.

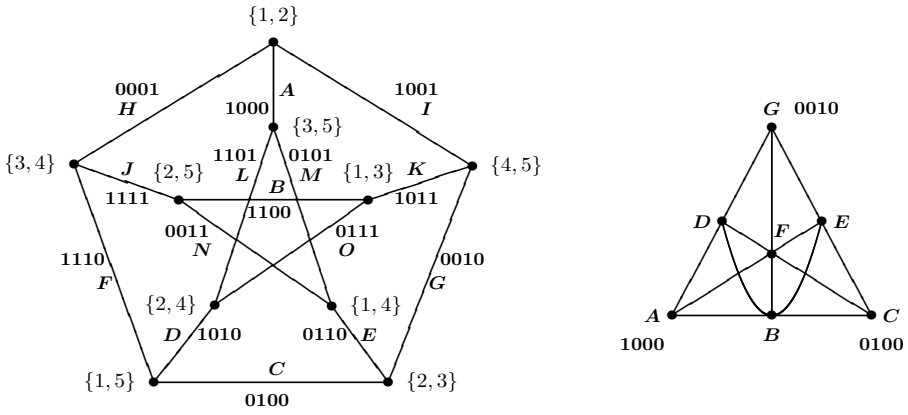

Figure 1: Petersen graph $\mathcal{O}_2$ and Fano plane $\mathcal{F} = (0,0,0,1)'$

- Let σ be any 8-cycle in Γ and denote the edges of Γ that are not in σ by $A, \ldots, G$. Label the type II line (from the spokes) on Γ with A, B, C.
- The two lines in $\mathcal{F}$ that are of type I must pass through one of A, B, C, so suppose it is C. Then D, F and E, G on Γ are at the two vertices of the edge labelled C. We can place D, F arbitrarily but then E, G must be placed according to the type III lines, by finding the 6-cycle through D and E, say. This places all the points of $\mathcal{F}$ on Γ.
- To fit the remaining points of $PG_3(\mathbb{F}_2)$ we can repeat what we have just done with another 8-cycle, or we can introduce coordinates: if $\mathcal{F}$ is the plane $(0,0,0,1)'$, and points $A, \ldots, G$ as shown in Fig. 1, then any one of the remaining eight points of $PG_3(\mathbb{F}_2)$ can be inserted randomly on Γ, and the remaining correspondences will follow by employing the rules for type I lines. In Fig. 1 we chose $H = (0,0,0,1)$ to label the edge $[\{1,2\}, \{3,4\}]$.

One can check that the complement in the edge set of any 8-cycle is indeed a plane $PG_2(\mathbb{F}_2)$: e.g. the 8-cycle with consecutive edges $AMEGKBJH$ has complement $(1,0,1,1)'$.

Conversely, to construct the Petersen graph from $PG_3(\mathbb{F}_2)$ we want a set $\mathcal{L}$ of ten lines such that each point of $PG_3(\mathbb{F}_2)$ is on exactly two of these lines, or alternatively, each Fano plane in $PG_3(\mathbb{F}_2)$ contains exactly two lines of $\mathcal{L}$. This can be done as follows:

- denote the 15 points of $PG_3(\mathbb{F}_2)$ by $\{A, B, \ldots, O\}$ and suppose that ABC is a line, π_A, π_B, π_C the three planes containing it, with point sets

$$\pi_A : \{A,B,C,D,E,F,G\}; \pi_B : \{A,B,C,H,I,J,K\}; \pi_C : \{A,B,C,L,M,N,O\};$$

- excluding ABC, take the other two lines through A in π_A, B in π_B, and C in π_C, respectively, writing $a = ADE, b = AFG, c = BHI, d = BJK, e = CLM, f = CNO$;
- let DHL be a line skew to ABC and let Π_D, Π_H, Π_L be the three planes containing DHL, and suppose that $A \in \Pi_D, B \in \Pi_H, C \in \Pi_L$, and so $ADE = \pi_A \cap \Pi_D, BHI = \pi_B \cap \Pi_H, CLM = \pi_C \cap \Pi_L$;
- let g be the third line through D in Π_D, h that through H in Π_H, and i that through L in Π_L; suppose $g \cap \pi_B = K, g \cap \pi_C = N$, $h \cap \pi_A = F, h \cap \pi_C = O$, $i \cap \pi_A = G, i \cap \pi_B = J$, so $g = DKN, h = HFO, i = LGJ$;
- for the final (10^{th}) line take $j = EIM$. (That this is a line is easy to see: suppose, as vectors in $\mathbb{F}_2^4$, $A = v_1, B = v_2, C = v_1 + v_2, D = w_1, H = w_2, L = w_1 + w_2$, where these vectors are all distinct. Then $E = v_1 + w_1, I = v_2 + w_2, M = v_1 + v_2 + w_1 + w_2$, which shows that EIM is a line.)

- The ten lines are $a = ADE, b = AFG, c = BHI, d = BJK, e = CLM, f = CNO, g = DKN, h = HFO, i = LGJ, j = EIM$.

These lines are taken to be the ten vertices, with $\{A, \ldots, O\}$ as the set of 15 edges, and vertices are adjacent if the triples intersect. So $A = ab, B = cd$ and so on. We have a 3-regular connected graph and it can be fitted onto the Petersen diagram as shown in Figure 2.

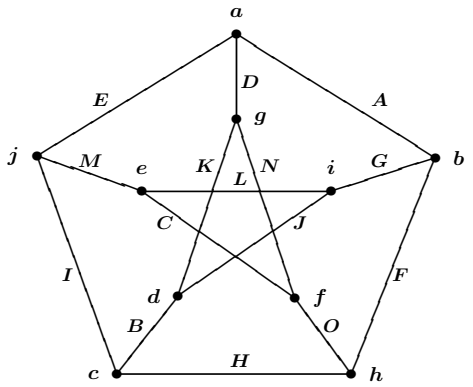

Figure 2: $PG_3(\mathbb{F}_2)$ to Petersen graph

Using coordinates, we can choose $A = (1,0,0,0)$, $B = (0,1,0,0)$, $D = (0,0,1,0)$, $H = (0,0,0,1)$ and then C, E, L, I, M will follow. Then we can choose $K = (1,0,0,1)$, so that N follows, and choosing $F = (0,1,1,0)$, all the remaining edges and vertices follow. One can check that the complement of any of the planes, e.g. Π_D, is an 8-cycle.

7 Permutation decoding

In [19, Lemma 7] the following result, which holds for any information set, was proved:

Result 5. *Let C be a linear code with minimum weight d, $\mathcal{I}$ an information set, $\mathcal{C}$ the corresponding check set and $\mathcal{P} = \mathcal{I} \cup \mathcal{C}$. Let G be an automorphism group of C, and n the maximum value of $|\mathcal{O} \cap \mathcal{I}|/|\mathcal{O}|$, over the G-orbits $\mathcal{O}$. If $s = \min(\lceil \frac{1}{n} \rceil - 1, \lfloor \frac{d-1}{2} \rfloor)$, then G is an s-PD-set for C.*

If the group G is transitive then $|\mathcal{O}|$ is the degree of the group and $|\mathcal{O} \cap \mathcal{I}|$ is the dimension of the code. This is applicable to codes from incidence matrices of connected regular graphs with automorphism groups transitive on edges, leading to the following result from [9]:

Result 6. *Let $\Gamma = (V, E)$ be a regular graph of valency k with automorphism group A transitive on edges. Let M be an incidence matrix for Γ. If, for p a prime, $C = C_p(M)$ is a $[|E|, |V| - \varepsilon, k]_p$ code, where $\varepsilon \in \{0, 1, \ldots, |V| - 1\}$, then any transitive subgroup of A will serve as a PD-set for full error correction for C.*

Using the hull, more errors can be corrected, as shown in [12, Corollary 4]. For the binary hulls of the graphs $\mathcal{O}_k$, since $\mathrm{Aut}(\mathcal{O}_k)$ acts transitively on edges for all $k \geq 2$, we have the following:

Proposition 5. *Let $\Gamma = (V, E) = \mathcal{O}_k$, $k \geq 2$, $A \subseteq \mathrm{Aut}(\mathcal{O}_k)$ transitive on edges, G_k an incidence matrix for Γ and $H_k = \mathrm{Hull}(C_2(G_k))$. Then A can be used as a k-PD-set for H_k.*

Proof: We use Result 5. By Result 4, if $\dim(H_k) = d_H$, then $d_H = \binom{2k-1}{k} + 2^{k-1} - 1$ for k even, and $d_H = \binom{2k}{k-1} - 1$ for k odd.

By Proposition 3, the minimum weight of H_k is at least $2k + 2$, so it can correct k errors. In the notation of Result 5, if $s = \min(\lceil \frac{1}{n} \rceil - 1, \lfloor \frac{d-1}{2} \rfloor) = \min(\lceil \frac{|E|}{d_H} \rceil - 1, k)$, then s errors can be corrected by

using A as a PD-set. We need to show that $s = k$. This will be so if $|E|/d_H - 1 \geq k$, i.e. $|E| \geq d_H(k+1)$, so if $|V| \geq 2d_H$. This is not difficult to prove in the two cases, k even or k odd. □

For smaller, and thus more efficient, PD-sets one needs first to find suitable information sets and then search for suitable sets of elements from the automorphism group. For example, for $k = 2$ when $H_2 = \mathrm{Hull}(C_2(G_2)) = \mathcal{H}_4^{\perp}$, small PD-sets were found in [11]. Also, a PD-set of five elements for H_2, thus attaining the Gordon-Schönheim bound was found in [25]. By computation with Magma we also found small PD-sets for full error correction for some of the other binary hulls. Explicit s-PD-sets that satisfy the Gordon-Schönheim bound for s-PD-sets for the class of q-ary simplex codes (the duals of the Hamming codes) for all prime powers q can be found in [11].

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