Codes, Designs and Graphs from the Janko Groups J_1 and J_2

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Abstract

We construct some codes, designs and graphs that have the first or second Janko group, J_1 or J_2, respectively, acting as an automorphism group. We show computationally that the full automorphism group of the design or graph in each case is J_1, J_2 or $\bar{J}_2$, the extension of J_2 by its outer automorphism, and we show that for some of the codes the same is true.

1 Introduction

Error-correcting codes that have large automorphism groups whose properties are extensively studied can be useful in applications as the group can help in determining the code's properties, and can be useful in decoding algorithms: see Huffman [8] for a discussion of possibilities, including the question of the use of permutation decoding by searching for PD-sets.

We consider here the primitive representations of the simple Janko groups J_1 and J_2, as described, for example, in [6]. For each group, using Magma [3], we constructed designs and graphs that have the group acting primitively on points as automorphism group, and, for a selection of

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small primes, codes over that prime field derived from the designs or graphs that also have the group acting as automorphism group. In each case we found codes with good parameters. For each code, the code automorphism group at least contains the associated Janko group, but in fact we did not find any in which it was bigger (unless the code was the full space, or of codimension 1 in the full space).

A sample of our results is attached in the appendix, but the full set can be obtained at a web site (address given in the text). The appendix includes the Magma code that was used to obtain the designs and codes. Some small but interesting cases where we could determine the full automorphism group of the code and the full weight enumerator, are given in Section 6.

We looked into the possibility of finding strongly regular graphs amongst the graphs obtained from this primitive action, although such a study has been conducted independently elsewhere.

2 Terminology and notation

Our notation will be standard, and as in [1]. An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a t-(v, k, λ) design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points, and every t distinct points are together incident with precisely λ blocks. The **dual** structure of $\mathcal{D}$ is $\mathcal{D}^t = (\mathcal{B}, \mathcal{P}, \mathcal{I})$. Thus the transpose of an incidence matrix for $\mathcal{D}$ is an incidence matrix for $\mathcal{D}^t$. We will say that the design is **symmetric** if it has the same number of points and blocks, and **self-dual** if it is isomorphic to its dual.

The code C_F of the design $\mathcal{D}$ over the finite field F is the space spanned by the incidence vectors of the blocks over F. We take F to be a prime field F_p, in which case we write also C_p for C_F, and refer to the dimension of C_p as the **p-rank** of $\mathcal{D}$. In the general case of a 2-design, the prime must divide the order of the design, i.e. $r - \lambda$, where r is the replication number for the design, that is, the number of blocks through a point. If the point set of $\mathcal{D}$ is denoted by $\mathcal{P}$ and the block set by $\mathcal{B}$, and if $\mathcal{Q}$ is any subset of $\mathcal{P}$, then we will denote the incidence vector of $\mathcal{Q}$ by $v^{\mathcal{Q}}$. Thus $C_F = \langle v^B \mid B \in \mathcal{B} \rangle$, and is a subspace of $F^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to F.

For any code C of length n over the field F, the **dual** or **orthogonal** code $C^{\perp}$ is the orthogonal under the standard inner product, i.e. $C^{\perp} = \{v \in F^n | (v, c) = 0 \text{ for all } c \in C\}$. A code C is **self-orthogonal** if $C \subseteq C^{\perp}$ and is **self-dual** if $C = C^{\perp}$. The **hull** of a design's code over some field is the intersection $C \cap C^{\perp}$. If a linear code over a field of order q is of length n, dimension k, and minimum weight d, then we write $[n, k, d]_q$ to show this information. If c is a codeword then the **support** of c is the set of non-zero coordinate positions of c. A **constant word** in the code is a codeword, all of whose coordinate entries are either 0 or 1. The all-one vector will be denoted by $\jmath$, and is the constant vector of weight the length of the code. Two linear codes of the same length and over the same field are **equivalent** if each can be obtained from the other by permuting the coordinate positions and multiplying each coordinate position by a non-zero field element. They are **isomorphic** if they can be obtained from one another by permuting the coordinate positions. An **automorphism** of a code C is any permutation of the coordinate positions that maps codewords to codewords. An automorphism thus preserves each weight class of C. Notice that if C is the code from an incidence structure, then any automorphism of the incidence structure will induce an automorphism of the code; the converse need not be true.

Terminology for graphs is standard: our graphs are undirected, the **valency** of a vertex is the number of edges containing the vertex; the **girth** of a graph is the number of edges in the smallest cycle in the graph, and the **diameter** of a graph is the length of the longest path in the graph. A graph is **regular** if all the vertices have the same valence, and a regular graph is **strongly regular** of type (n, k, λ, μ) if it has n vertices, valence k, and if any two adjacent vertices are together adjacent to λ vertices, while any two non-adjacent vertices are together adjacent to μ vertices.

3 The construction

Our computations are based on the following construction:

Proposition 1 *Let G be a finite primitive permutation group acting on the set Ω of size n. Let $\alpha \in \Omega$, and let $\Delta \neq \{\alpha\}$ be an orbit of the stabilizer G_α of α. If*

$$\mathcal{B} = \{\Delta^g : \ g \in G\}$$

and, given $\delta \in \Delta$,

$$\mathcal{E} = \{\{\alpha, \delta\}^g \ : \ g \in G\},$$

then $\mathcal{B}$ forms a self-dual 1-$(n, |\Delta|, |\Delta|)$ design with n blocks, and $\mathcal{E}$ forms the edge set of a regular connected graph of valency $|\Delta|$, with G acting as an automorphism group on each of these structures, primitive on vertices of the graph, and on points and blocks of the design.

Proof: We have $|G| = |\Delta^G||G_\Delta|$, and clearly $G_\Delta \supseteq G_\alpha$. Since G is primitive on Ω, G_α is maximal in G, and thus $G_\Delta = G_\alpha$, and $|\Delta^G| = |\mathcal{B}| = n$. This proves that we have a 1-$(n, |\Delta|, |\Delta|)$ design.

For the graph we notice that the vertices adjacent to α are the vertices in Δ. Now as we orbit these pairs under G, we get the nk ordered pairs, and thus $nk/2$ edges, where $k = \Delta$. Since the graph has G acting, it is clearly regular, and thus the valency is k as required, i.e. the only vertices adjacent to α are those in the orbit Δ. The graph must be connected, as a maximal connected component will form a block of imprimitivity, contradicting the group's primitive action.

Now notice that an adjacency matrix for the graph is simply an incidence matrix for the 1-design, so that the 1-design is necessarily self-dual. This proves all our assertions. □

Note that if we form any union of orbits of the stabilizer of a point, including the orbit consisting of the single point, and orbit this under the full group, we will still get a self-dual symmetric 1-design with the group operating. Thus the orbits of the stabilizer can be regarded as "building blocks". Since the complementary design (i.e. taking the complements of the blocks to be the new blocks) will have exactly the same properties, we will assume that our block size is at most $v/2$.

In fact this will give us all possible designs on which the group acts primitively on points and blocks:

Lemma 2 *If the group G acts primitively on the points and the blocks of a symmetric* 1*-design $\mathcal{D}$, then the design can be obtained by orbiting a union of orbits of a point-stabilizer, as described in the proposition.*

Proof: Suppose that G acts primitively on points and blocks of the 1-(v, k, k) design $\mathcal{D}$. Let $\mathcal{B}$ be the block set of $\mathcal{D}$; then if B is any block of $\mathcal{D}$, $\mathcal{B} = B^G$. Thus $|G| = |\mathcal{B}||G_B|$, and since G is primitive, G_B is maximal and thus $G_B = G_\alpha$ for some point. Thus G_α fixes B, so this must be a union of orbits of G_α. □

It is well known, and easy to see, that if the group is rank-3, i.e. the stabilizer of a point has exactly three orbits, then the graph formed as described in Proposition 1 will be strongly regular. In case the group is not of rank 3, this might still happen, and we examined this question also.

We took G to be J_1 and J_2, the first and second Janko groups, respectively: see [9]. Note that J_1 has no outer automorphisms, and thus is its own automorphism group, whereas J_2 has an involutary outer automorphism, so its automorphism group, which we will denote by $\bar{J}_2$, is a split extension of J_2 by Z_2, with double the order.

We looked first at J_1, which is of order 175560, and its maximal subgroups and primitive permutation representations via the coset action on these subgroups: see [6, 7]. There are seven distinct primitive representation, of degree 266, 1045, 1463, 1540, 1596, 2926, and 4180, respectively. We then looked at J_2, of order 604800, which has nine primitive representations, of degree 100, 280, 315, 525, 840, 1008, 1800, 2016 and 10080, respectively. The last degree is a little large to compute with comfortably.

For each of these groups, using Magma [3], we found the designs and graphs as described in Proposition 1, and found the p-rank of the designs for some small set of values of the prime p. To aid in the classification, we also found the dimension of the hull of the design for each of these primes. We also looked for strongly regular graphs for each group, and found three for J_2, all of which are known: see Brouwer and van Lint [4], and Brouwer [5]. We then also took a closer look at some of the more interesting codes that arose, asking what the basic coding properties were, and if the full automorphism group could be established. We did not find any that had automorphism group properly between the symmetric group and J_1, J_2 or $\bar{J}_2$, but we were in fact only able to compute a few. Nevertheless, it seems likely that this might always be the case for these groups: see, however, the comment in Section 7. Our results are listed in Section 6.

4 The computations for J_1

For each of the seven primitive representations, using Magma, we constructed the permutation group and formed the orbits of the stabilizer of a point. For each of the non-trivial orbits, we formed the symmetric 1-design as described in Proposition 1. Note that because of the maximality of the point stabilizer, there is only the one orbit of length 1: for suppose G is the group, and suppose that G_α fixes also β. Then $G_\alpha = G_\beta$. Since G is transitive, there exists $g \in G$ such that $\alpha^g = \beta$. Then $(G_\alpha)^g = G_{\alpha^g} = G_\beta = G_\alpha$, and thus $g \in N_G(G_\alpha) = N$, the normalizer of G_α in G. Since G_α is maximal in G, we have $N = G$ or $N = G_\alpha$. But G is simple, so we must have $N = G_\alpha$, so that $g \in G_\alpha$ and so $\beta = \alpha$. Since these 1-designs do not by their parameters give any indication of what primes might give codes that are likely to be of any use either for applications or for characterization purposes, we took a small set of the lowest primes, i.e. $\{2, 3, 5, 7, 11\}$, and found the dimension of the code and its hull for each of these primes. Note also that since 19 is a divisor of the order of J_1, in some of the smaller cases it is worthwhile also to look at codes over the field of order 19. We also found the automorphism group of each design, which will be the same as the automorphism group of the regular graph. Where computationally possible we also found the automorphism group of the code.

Conclusions from our results are summarized below. In brief, we found that there are 245 designs formed in this manner from single orbits and that none of them is isomorphic to any other of the designs in this set. In every case the full automorphism group of the design or graph is J_1. In all but 34 of the designs, the dimensions of the code or the hull over the set of primes given above distinguished the designs. For the 34 remaining, these occurred in 17 pairs in which the set of dimensions for each pair was

Degree	#	length				
266	5	132	110	12	11	
1045	11	168(5)	56(3)	28	8	
1463	22	120(7)	60(9)	20(2)	15(2)	12
1540	21	114(9)	57(6)	38(4)	19	
1596	19	110(13)	55(2)	22(2)	11	
2926	67	60(34)	30(27)	15(5)		
4180	107	42(95)	21(6)	14(4)	7	

Table 1: Orbits of a point-stabilizer of J_1

identical, but distinct from all the other pairs. For these we tried a few more primes but got no distinction, possibly, but not necessarily, indicating that the codes are isomorphic. This could not be tested with Magma as the codes have block length that is too long. However, for these 17 pairs we simply used Magma to test the design isomorphism, and thus obtained the stated conclusion.

In Table 1, the first column gives the degree, the second the number of orbits, and the remaining columns give the length of the orbits of length greater than 1, with the number of that length in parenthesis behind the length in case there is more than one of that length. The pairs that had the same code dimensions occurred as follows: for degrees 266, 1045 and 1596, there were no such pairs; for degree 1463 there were two pairs, both for orbit size 60; for degree 1540, there were two pairs, for orbit size 57 and 114 respectively; for degree 2926 there was one pair for orbit size 60; for degree 4180 there were 12 pairs, for orbit size 42.

We make the general comment that for each one of these 245 designs (or graphs) there was at least one prime from our small set that gave an "interesting code", i.e. a code that is not the full space or of codimension 1. Full details of the numbers obtained can be found at the web site:

`http://www.ces.clemson.edu/~keyj`

under the file "Janko groups and designs". We include a few sample results and the related Magma code in the appendix.

In summary then, we have the following:

Proposition 3 *If G is the first Janko group J_1, there are precisely* 245 *non-isomorphic self-dual* 1*-designs obtained by taking all the images under G of the non-trivial orbits of the point stabilizer in any of G's primitive representations, and on which G acts primitively on points and blocks. In each case the full automorphism group is J_1. Every primitive action on*

Degree	#	length						
100	3	63	36					
280	4	135	108	36				
315	6	160	80	32(2)	10			
525	6	192(2)	96	32	12			
840	7	360	240	180	24	20	15	
1008	11	300	150(2)	100(2)	60(2)	50	25	12
1800	18	336	168(6)	84(3)	42(3)	28	21	14(2)
2016	18	300(2)	150(6)	75(5)	50(2)	25	15	

Table 2: Orbits of a point-stabilizer of J_2 (of degree ≤ 2016)

symmetric 1-designs can be obtained by taking the union of such orbits and orbiting under G.

We tested the graphs for strong regularity in the cases of the smaller degree, and did not find any that were strongly regular. We have not however, as yet, tested all the graphs. We also found the designs and their codes for some of the unions of orbits in some cases. We found that some of the codes were the same for some primes, but not for all.

5 The computations for J_2

This group has nine primitive representations, as already mentioned, but we did not compute with the largest degree. Thus our results cover only the first eight. Our results for J_2 are different from those for J_1, due to the existence of an outer automorphism. The main difference is that usually the full automorphism group is $\bar{J}_2$, and that in the cases where it was only J_2, there would be another orbit of that length that would give an isomorphic design, and which, if the two orbits were joined, would give a design of double the block size and automorphism group $\bar{J}_2$. A similar conclusion held if some union of orbits was taken as a base block.

From these eight primitive representations, we obtained in all 51 non-isomorphic symmetric designs on which J_2 acts primitively. Table 2 gives the same information for J_2 that Table 1 gives for J_1. The automorphism group of the design in each case was J_2 or $\bar{J}_2$. Where J_2 was the full group, there is another copy of the design for another orbit of the same length. This occurred in the following cases: degree 315, orbit length 32; degree 1008, orbit lengths 60, 100 and 150; degree 1800, orbit lengths 42, 42, 84 and 168; degree 2016, orbit lengths 50, 75, 75, 150, 150, and 300. We note again that the p-ranks of the design and their hulls gave an initial indication of possible isomorphisms and clear non-isomorphisms, so that only the few mentioned needed be tested. This reduced the computations tremendously.

We also found three strongly regular graphs (all of which are known: see Brouwer [5]): that of degree 100 from the rank-3 action, of course, and two more of degree 280 from the orbits of length 135 and 36, giving strongly regular graphs with parameters (280,135,70,60) and (280,36,8,4) respectively. The full automorphism group is $\bar{J}_2$ in each case. We have not checked all the other representations but note that this is the only one with point stabilizer having exactly four orbits. Note that Bagchi [2] found a strongly regular graph with J_2 acting.

6 Automorphism groups of the codes

Clearly the automorphism group of any of the codes will contain the automorphism group of the design from which it is formed. We looked at some of the codes that were computationally feasible to find out if the groups J_1 and $\bar{J}_2$ formed the full automorphism group in any of the cases when the code was not the full vector space. We first mention the following lemma:

Lemma 4 *Let C be the linear code of length n of an incidence structure $\mathcal{I}$ over a field F. Then the automorphism group of C is the full symmetric group if and only if $C = F^n$ or $C = F\boldsymbol{\jmath}^{\perp}$.*

Proof: Suppose $\mathrm{Aut}(C)$ is S_n. C is spanned by the incidence vectors of the blocks of $\mathcal{I}$; let B be such a block and suppose it has k points, and so it gives a vector of weight k in C. Clearly C contains the incidence vector of any set of k points, and thus, by taking the difference of two such vectors that differ in just two places, we see that C contains all the vectors of weight 2 having as non-zero entries 1 and -1. Thus $C = F\boldsymbol{\jmath}^{\perp}$ or F^n. The converse is clear. $\square$

Huffman [8] has more on codes and groups, and in particular, on the possibility of the use of permutation decoding for codes with large groups acting. See also Knapp and Schmid [10] for more on codes with prescribed groups acting.

Most of the codes we looked at were too large to find the automorphism group, but we did find, through computation with Magma, the list given below. Note that we could in some cases look for the full group of the hull, and from that deduce the group of the code, since $\mathrm{Aut}(C) = \mathrm{Aut}(C^{\perp}) \subseteq \mathrm{Aut}(C \cap C^{\perp})$. In the few cases where the group of the design was known only to be the simple group J_2, we could construct the extended group $\bar{J}_2$ and check to see if the basis of the code of the design was mapped into the code by the outer automorphism. In all cases we did get the full extended group.

In each of the following we consider the primitive action of J_1 or J_2 on a design formed as described in Proposition 1 from an orbit or a union of orbits, and the codes are the codes of the associated 1-design.

Full automorphism groups of the codes

1. For J_2 of degree 100, $\bar{J}_2$ is the full automorphism group of the design with parameters 1-$(100, 36, 36)$, and it is the automorphism group of the self-orthogonal doubly-even $[100, 36, 16]_2$ binary code of this design.

2. For J_2 of degree 280, $\bar{J}_2$ is the full automorphism group of the design with parameters 1-$(280, 108, 108)$, and it is the automorphism group of the self-orthogonal doubly-even $[280, 14, 108]_2$ binary code of this design. The weight distribution of this code is

   ```
    <0, 1>, <108, 280>, <128, 1575>, <136, 2520>, <140, 7632>,
   <144, 2520>, <152, 1575>, <172, 280>, <280, 1>
   ```

   Thus the words of minimum weight (i.e. 108) are the incidence vectors of the design.

3. For J_2 of degree 315, $\bar{J}_2$ is the full automorphism group of the design with parameters 1-$(315, 64, 64)$ (by taking the union of the two orbits of length 32), and it is the automorphism group of the self orthogonal doubly-even $[315, 28, 64]_2$ binary code of this design. The weight distribution of the code is as follows:

   ```
   <0, 1>,<64, 315>,<96, 6300>,<104, 25200>,<112, 53280>,
   <120, 242760>,<124, 201600>,<128, 875700>,<132, 1733760>,
   <136, 4158000>,<140, 5973120>,<144, 12626880>,<148, 24232320>,
   <152, 35151480>,<156, 44392320>,<160, 53040582>,
   <164, 41731200>,<168, 28065120>,<172, 13023360>,<176, 2129400>,
   <180, 685440>,<184, 75600>,<192, 10710>,<200, 1008>
   ```

   Thus the words of minimum weight (i.e. 64) are the incidence vectors of the blocks of the design.

   Furthermore, the designs from the two orbits of length 32 in this case, i.e. 1-$(315, 32, 32)$ designs, each have J_2 as their automorphism group. Their binary codes are equal, and are $[315, 188]_2$ codes, with hull the 28-dimensional code described above. The automorphism group of this 188-dimensional code is again $\bar{J}_2$. The minimum weight is at most 32. This is also the binary code of the design from the orbit of length 160.

4. For J_2 of degree 315, $\bar{J}_2$ is the full automorphism group of the design with parameters 1-$(315, 160, 160)$ and it is the automorphism group of the $[315, 265]_5$ 5-ary code of this design. This code is also the 5-ary code of the design obtained from the orbit of length 10, and from that of the orbit of length 80, so we can deduce that the minimum weight is at most 10. The hull is a $[315, 15, 155]_5$ code and again with $\bar{J}_2$ as full automorphism group.

5. For J_2 of degree 315, $\bar{J}_2$ is the full automorphism group of the design with parameters 1-$(315, 80, 80)$ from the orbit of length 80, and it is the automorphism group of the self-orthogonal doubly-even $[315, 36, 80]_2$ binary code of this design. The minimum words of this code are precisely the 315 incidence vectors of the blocks of the design.

7 Deductions

We could use these computations to conjecture that the automorphism groups of the designs obtained in this way from a primitive representation of a simple group G will have the automorphism group $\mathrm{Aut}(G)$ as its automorphism group, unless the design is isomorphic to another one constructed in this way, in which case the automorphism group of the design will be a proper subgroup of the $\mathrm{Aut}(G)$ containing G.

For the automorphism group of the codes, we have not found a code that has automorphism group bigger than J_1 or $\bar{J}_2$ but not equal to the full symmetric group. However, examples certainly do exist where the automorphism group of the code is bigger than that of the design, but still not the full symmetric group: if $\mathcal{D}$ is the 2-(28,4,1) hermitian unital, the automorphism group is the unitary group $P\Gamma U_3(F_9)$, while its binary code, a $[28, 21, 4]_2$, has automorphism group the symplectic group $Sp_6(2)$ (see [1, page 301] for a detailed discussion).

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8 Appendix

```
Some results for J1
Loading "/applic/magma2.4
/libs/simgps/simgps"
 g:=SimGroup("J1");
 re:=SimRecord("J1");
 ma:=re`Max;
//J1 of degree 266

 g1:=ma[1];
 a1,a2,a3:=CosetAction(g,g1);
 pr:=[2,3,5,7,11];
 st:=Stabilizer(a2,1);
 orbs:=Orbits(st);#orbs;
5
lo:=[#orbs[i]: i in [1..#orbs]];
 lo;
[ 1, 132, 110, 11, 12 ]
 v:=Index(a2,st);
 for j:=2 to #lo do
 "orbs no",j,"of length",
 #orbs[j];
 edg:={1,Setseq(orbs[j])[1]}^a2;
 gr:=Graph<v|edg>;
 "no of edges=",#edg;
 "valence=",Valence(gr),
 "girth=",Girth(gr),
 "diameter=",Diameter(gr);
 Order(AutomorphismGroup(gr));
 blox:=Setseq(orbs[j]^a2);
 des:=Design<1,v|blox>;
 Order(AutomorphismGroup(des));
  for i:=1 to 5 do
  p:=pr[i];
  dc:=LinearCode(des,GF(p));
  d1:=Dim(dc);d2:=Dim(Dual(dc));
 d3:=Dim(dc meet Dual(dc));
 p,"dim=",d1,
 "dual=",d2,"dh=",d3;
 end for;
 end for;
orbs no 2 of length 132
no of edges= 17556
valence= 132 girth= 3 diameter= 2
175560
175560
 2 dim= 112 dual= 154 hull= 0
 3 dim= 153 dual= 113 hull= 0
 5 dim= 266 dual= 0 hull= 0
 7 dim= 266 dual= 0 hull= 0
 11 dim= 209 dual= 57 hull= 0
orbs no 3 of length 110
no of edges= 14630
valence= 110 girth= 3 diameter= 2
175560
175560
 2 dim= 188 dual= 78 hull= 0
 3 dim= 266 dual= 0 hull= 0
 5 dim= 56 dual= 210 hull= 56
 7 dim= 266 dual= 0 hull= 0
 11 dim= 209 dual= 57 hull= 0
orbs no 4 of length 11
no of edges= 1463
valence= 11 girth= 5 diameter= 4
175560
175560
 2 dim= 190 dual= 76 hull= 0
 3 dim= 266 dual= 0 hull= 0
 5 dim= 266 dual= 0 hull= 0
 7 dim= 266 dual= 0 hull= 0
 11 dim= 209 dual= 57 hull= 0
orbs no 5 of length 12
no of edges= 1596
valence= 12 girth= 5 diameter= 3
175560
175560
 2 dim= 112 dual= 154 hull= 0
 3 dim= 153 dual= 113 hull= 0
 5 dim= 266 dual= 0 hull= 0
 7 dim= 266 dual= 0 hull= 0
 11 dim= 266 dual= 0 hull= 0
---------------------------------
//J1 of degree 1045

orbs:=Orbits(st);#orbs;
11
[ 1, 28, 168, 168, 168, 56, 168,
168, 56, 56, 8 ]
orbs no 2 of length 28
no of edges= 14630
valence= 28 girth= 3 diameter= 3
175560
```

```
175560
 2 dim= 572 dual= 473 hull= 0
 3 dim= 1045 dual= 0 hull= 0
 5 dim= 1045 dual= 0 hull= 0
 7 dim= 770 dual= 275 hull= 0
 11 dim= 962 dual= 83 hull= 49
orbs no 3 of length 168
no of edges= 87780
valence= 168 girth= 3 diameter= 3
175560
175560
 2 dim= 856 dual= 189 hull= 0
 3 dim= 702 dual= 343 hull= 0
 5 dim= 1045 dual= 0 hull= 0
 7 dim= 684 dual= 361 hull= 89
 11 dim= 1045 dual= 0 hull= 0
orbs no 4 of length 168
no of edges= 87780
valence= 168 girth= 3 diameter= 2
175560
175560
 2 dim= 968 dual= 77 hull= 0
 3 dim= 360 dual= 685 hull= 0
 5 dim= 856 dual= 189 hull= 56
 7 dim= 924 dual= 121 hull= 0
 11 dim= 968 dual= 77 hull= 0
orbs no 5 of length 168
no of edges= 87780
valence= 168 girth= 3 diameter= 2
175560
175560
 2 dim= 758 dual= 287 hull= 134
 3 dim= 702 dual= 343 hull= 0
 5 dim= 1045 dual= 0 hull= 0
 7 dim= 791 dual= 254 hull= 0
 11 dim= 1045 dual= 0 hull= 0
orbs no 6 of length 56
no of edges= 29260
valence= 56 girth= 3 diameter= 3
175560
175560
 2 dim= 968 dual= 77 hull= 0
 3 dim= 1045 dual= 0 hull= 0
 5 dim= 1045 dual= 0 hull= 0
 7 dim= 924 dual= 121 hull= 0
 11 dim= 1045 dual= 0 hull= 0
orbs no 7 of length 168
no of edges= 87780
valence= 168 girth= 3 diameter= 2
175560
175560
 2 dim= 856 dual= 189 hull= 0
 3 dim= 493 dual= 552 hull= 133
 5 dim= 912 dual= 133 hull= 56
 7 dim= 924 dual= 121 hull= 0
 11 dim= 1045 dual= 0 hull= 0
orbs no 8 of length 168
no of edges= 87780
valence= 168 girth= 3 diameter= 3
175560
175560
 2 dim= 758 dual= 287 hull= 134
 3 dim= 702 dual= 343 hull= 0
 5 dim= 1045 dual= 0 hull= 0
 7 dim= 684 dual= 361 hull= 89
 11 dim= 968 dual= 77 hull= 0
orbs no 9 of length 56
no of edges= 29260
valence= 56 girth= 3 diameter= 3
175560
175560
 2 dim= 968 dual= 77 hull= 0
 3 dim= 836 dual= 209 hull= 133
 5 dim= 1045 dual= 0 hull= 0
 7 dim= 924 dual= 121 hull= 0
 11 dim= 968 dual= 77 hull= 0
orbs no 10 of length 56
no of edges= 29260
valence= 56 girth= 4 diameter= 3
175560
175560
 2 dim= 512 dual= 533 hull= 152
 3 dim= 1045 dual= 0 hull= 0
 5 dim= 1045 dual= 0 hull= 0
 7 dim= 924 dual= 121 hull= 0
 11 dim= 1045 dual= 0 hull= 0
orbs no 11 of length 8
no of edges= 4180
valence= 8 girth= 5 diameter= 5
175560
175560
 2 dim= 968 dual= 77 hull= 0
```

```
 3 dim= 836 dual= 209 hull= 133
 5 dim= 1045 dual= 0 hull= 0
 7 dim= 1045 dual= 0 hull= 0
 11 dim= 1045 dual= 0 hull= 0
---------------------------------

Some results for J_2:
//J2 of degree 100

no. of orbs= 3
[ 1, 63, 36 ]
degree= 100
orbs no 2 of length 63
no of edges= 3150
valence= 63 girth= 3 diameter= 2
1209600
1-(100, 63, 63) Design with
100 blocks
1209600
 2 dim= 100 dual= 0 hull= 0
 3 dim= 36 dual= 64 hull= 0
 5 dim= 100 dual= 0 hull= 0
 7 dim= 63 dual= 37 hull= 0
 11 dim= 100 dual= 0 hull= 0
orbs no 3 of length 36
no of edges= 1800
valence= 36 girth= 3 diameter= 2
1209600
1-(100, 36, 36) Design with
100 blocks
1209600
 2 dim= 36 dual= 64 hull= 36
 3 dim= 63 dual= 37 hull= 0
 5 dim= 100 dual= 0 hull= 0
 7 dim= 100 dual= 0 hull= 0
 11 dim= 100 dual= 0 hull= 0
--------------------------------
//J2 of degree 280

Order = 604800
no. of orbs= 4
[ 1, 135, 108, 36 ]
degree= 280
orbs no 2 of length 135
no of edges= 18900
valence= 135 girth= 3 diameter= 2
1209600
1-(280, 135, 135) Design with
280 blocks
1209600
 2 dim= 280 dual= 0 hull= 0
 3 dim= 216 dual= 64 hull= 0
 5 dim= 42 dual= 238 hull= 42
 7 dim= 280 dual= 0 hull= 0
 11 dim= 280 dual= 0 hull= 0
orbs no 3 of length 108
no of edges= 15120
valence= 108 girth= 3 diameter= 2
1209600
1-(280, 108, 108) Design with
280 blocks
1209600
 2 dim= 14 dual= 266 hull= 14
 3 dim= 216 dual= 64 hull= 0
 5 dim= 280 dual= 0 hull= 0
 7 dim= 280 dual= 0 hull= 0
 11 dim= 280 dual= 0 hull= 0
orbs no 4 of length 36
no of edges= 5040
valence= 36 girth= 3 diameter= 2
1209600
1-(280, 36, 36) Design with
280 blocks
1209600
 2 dim= 62 dual= 218 hull= 62
 3 dim= 279 dual= 1 hull= 0
 5 dim= 280 dual= 0 hull= 0
 7 dim= 280 dual= 0 hull= 0
 11 dim= 280 dual= 0 hull= 0
---------------------------------
//J2 of degree 315

no. of orbs= 6
[ 1, 80, 160, 10, 32, 32 ]
degree= 315
orbs no 2 of length 80
no of edges= 12600
valence= 80 girth= 3 diameter= 2
1209600
1-(315, 80, 80) Design with
315 blocks
1209600
```

```
 2 dim= 36 dual= 279 hull= 36
 3 dim= 315 dual= 0 hull= 0
 5 dim= 265 dual= 50 hull= 15
 7 dim= 315 dual= 0 hull= 0
 11 dim= 315 dual= 0 hull= 0
orbs no 3 of length 160
no of edges= 25200
valence= 160 girth= 3 diameter= 2
1209600
1-(315, 160, 160) Design with
315 blocks
1209600
 2 dim= 188 dual= 127 hull= 28
 3 dim= 279 dual= 36 hull= 0
 5 dim= 265 dual= 50 hull= 15
 7 dim= 155 dual= 160 hull= 36
 11 dim= 279 dual= 36 hull= 0
orbs no 4 of length 10
no of edges= 1575
valence= 10 girth= 3 diameter= 4
1209600
1-(315, 10, 10) Design with
315 blocks
1209600
 2 dim= 154 dual= 161 hull= 0
 3 dim= 225 dual= 90 hull= 0
 5 dim= 265 dual= 50 hull= 15
 7 dim= 315 dual= 0 hull= 0
 11 dim= 315 dual= 0 hull= 0
orbs no 5 of length 32
no of edges= 5040
valence= 32 girth= 3 diameter= 3
604800
1-(315, 32, 32) Design with
315 blocks
604800
 2 dim= 188 dual= 127 hull= 28
 3 dim= 315 dual= 0 hull= 0
 5 dim= 315 dual= 0 hull= 0
 7 dim= 315 dual= 0 hull= 0
 11 dim= 301 dual= 14 hull= 0
orbs no 6 of length 32
no of edges= 5040
valence= 32 girth= 3 diameter= 3
604800
1-(315, 32, 32) Design with
315 blocks
604800
 2 dim= 188 dual= 127 hull= 28
 3 dim= 315 dual= 0 hull= 0
 5 dim= 315 dual= 0 hull= 0
 7 dim= 315 dual= 0 hull= 0
 11 dim= 301 dual= 14 hull= 0
> des5:
=Design<1,v|Setseq(orbs[5]^a2)>;
> des6:
=Design<1,v|Setseq(orbs[6]^a2)>;
>IsIsomorphic(des5,des6);
true
> bl:=orbs[5] join orbs[6];
> des56:=
Design<1,v|Setseq(bl^a2)>;
> des56;
1-(315, 64, 64) Design with
315 blocks
> #(AutomorphismGroup(des56));
1209600
----------------------------------
//J2 of degree 525

no. of orbs= 6
[ 1, 96, 192, 12, 192, 32 ]
degree= 525
orbs no 2 of length 96
no of edges= 25200
valence= 96 girth= 3 diameter= 2
1209600
1-(525, 96, 96) Design with
525 blocks
1209600
 2 dim= 98 dual= 427 hull= 62
 3 dim= 126 dual= 399 hull= 0
 5 dim= 525 dual= 0 hull= 0
 7 dim= 525 dual= 0 hull= 0
 11 dim= 525 dual= 0 hull= 0
orbs no 3 of length 192
no of edges= 50400
valence= 192 girth= 3 diameter= 2
1209600
1-(525, 192, 192) Design with
525 blocks
1209600
```

```
 2 dim= 112 dual= 413 hull= 112
 3 dim= 63 dual= 462 hull= 0
 5 dim= 525 dual= 0 hull= 0
 7 dim= 525 dual= 0 hull= 0
 11 dim= 525 dual= 0 hull= 0
orbs no 4 of length 12
no of edges= 3150
valence= 12 girth= 3 diameter= 4
1209600
1-(525, 12, 12) Design with
525 blocks
1209600
 2 dim= 364 dual= 161 hull= 0
 3 dim= 260 dual= 265 hull= 134
 5 dim= 365 dual= 160 hull= 90
 7 dim= 365 dual= 160 hull= 36
 11 dim= 365 dual= 160 hull= 0
orbs no 5 of length 192
no of edges= 50400
valence= 192 girth= 3 diameter= 2
1209600
1-(525, 192, 192) Design with
525 blocks
1209600
 2 dim= 112 dual= 413 hull= 112
 3 dim= 260 dual= 265 hull= 134
 5 dim= 365 dual= 160 hull= 90
 7 dim= 365 dual= 160 hull= 36
 11 dim= 365 dual= 160 hull= 0
orbs no 6 of length 32
no of edges= 8400
valence= 32 girth= 3 diameter= 4
1209600
1-(525, 32, 32) Design with
525 blocks
1209600
 2 dim= 160 dual= 365 hull= 0
 3 dim= 462 dual= 63 hull= 0
 5 dim= 525 dual= 0 hull= 0
 7 dim= 525 dual= 0 hull= 0
 11 dim= 525 dual= 0 hull= 0
> lo;
[ 1, 96, 192, 12, 192, 32 ]
> des3:=
Design<1,v|Setseq(orbs[3]^a2)>;
> des5:=
Design<1,v|Setseq(orbs[5]^a2)>;
> IsIsomorphic(des3,des5);
false
--------------------------------
//J2 of degree 840

no. of orbs= 7
[ 1, 180, 240, 20, 15, 360, 24 ]
degree= 840
orbs no 2 of length 180
no of edges= 75600
valence= 180 girth= 3 diameter= 2
1209600
1-(840, 180, 180) Design with
840 blocks
1209600
 2 dim= 280 dual= 560 hull= 120
 3 dim= 216 dual= 624 hull= 0
 5 dim= 517 dual= 323 hull= 42
 7 dim= 602 dual= 238 hull= 0
 11 dim= 602 dual= 238 hull= 0
orbs no 3 of length 240
no of edges= 100800
valence= 240 girth= 3 diameter= 2
1209600
1-(840, 240, 240) Design with
840 blocks
1209600
 2 dim= 120 dual= 720 hull= 120
 3 dim= 378 dual= 462 hull= 90
 5 dim= 692 dual= 148 hull= 42
 7 dim= 714 dual= 126 hull= 0
 11 dim= 714 dual= 126 hull= 0
orbs no 4 of length 20
no of edges= 8400
valence= 20 girth= 3 diameter= 4
1209600
1-(840, 20, 20) Design with
840 blocks
1209600
 2 dim= 350 dual= 490 hull= 230
 3 dim= 714 dual= 126 hull= 90
 5 dim= 692 dual= 148 hull= 42
 7 dim= 714 dual= 126 hull= 0
 11 dim= 714 dual= 126 hull= 0
orbs no 5 of length 15
```

```
no of edges= 6300
valence= 15 girth= 3 diameter= 4
1209600
1-(840, 15, 15) Design with
840 blocks
1209600
 2 dim= 680 dual= 160 hull= 0
 3 dim= 504 dual= 336 hull= 0
 5 dim= 692 dual= 148 hull= 42
 7 dim= 840 dual= 0 hull= 0
 11 dim= 840 dual= 0 hull= 0
orbs no 6 of length 360
no of edges= 151200
valence= 360 girth= 3 diameter= 2
1209600
1-(840, 360, 360) Design with
840 blocks
1209600
 2 dim= 350 dual= 490 hull= 266
 3 dim= 216 dual= 624 hull= 0
 5 dim= 307 dual= 533 hull= 132
 7 dim= 455 dual= 385 hull= 0
 11 dim= 455 dual= 385 hull= 0
orbs no 7 of length 24
no of edges= 10080
valence= 24 girth= 3 diameter= 3
1209600
1-(840, 24, 24) Design with
840 blocks
1209600
 2 dim= 350 dual= 490 hull= 266
 3 dim= 350 dual= 490 hull= 134
 5 dim= 455 dual= 385 hull= 90
 7 dim= 455 dual= 385 hull= 0
 11 dim= 455 dual= 385 hull= 0
> bl:=orbs[1] join orbs[7];
> blox:=Setseq(bl^a2);
> #blox;
840
> des17:=Design<1,v|blox>;
> des17;
1-(840, 25, 25) Design with
840 blocks
>Order(AutomorphismGroup(des17));
1209600
```

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