# The Limit Inferior and Limit Superior of a Sequence

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# Outline

### Definition

Let $(a_n)_{n \geq k}$ be a sequence of real numbers which is bounded. Also let $S = \{y : \exists (a_{n_p}) \subseteq (a_n) \ni a_{n_p} \to y\}$. Since $S$ is non empty by the Bolzano Weierstrass Theorem for Sequences, $\inf S$ and $\sup S$ both exist and are finite. We define

$$\begin{aligned} \liminf(a_n) &= \; = \underline{\lim}(a_n) = \text{ limit inferior } (a_n) = \inf S \\ \limsup(a_n) &= \; = \overline{\lim}(a_n) = \text{ limit superior } (a_n) = \sup S \end{aligned}$$

$S$ is called the set of subsequential limits of $(a_n)$.

### Example

For $((-1)^n)$, $S = \{-1, 1\}$ and $\underline{\lim}((-1)^n) = -1$ and $\overline{\lim}((-1)^n) = 1$.

**Example**

For $(\cos(n\pi/3))$, $S = \{-1, -1/2, 1/2, 1\}$ and $\underline{\lim}(a_n) = -1$ and $\overline{\lim}(a_n) = 1$.

**Example**

For $(\cos(n\pi/4))$, $S = \{-1, -1/\sqrt{2}, 0, 1/\sqrt{2}, 1\}$ and $\underline{\lim}(a_n) = -1$ and $\overline{\lim}(a_n) = 1$.

**Example**

For $(5 + \cos(n\pi/4))$, $S = \{5 - 1, 5 - 1/\sqrt{2}, 5 + 0, 5 + 1/\sqrt{2}, 5 + 1\}$ and $\underline{\lim}(a_n) = 4$ and $\overline{\lim}(a_n) = 6$.

## Theorem

*Let $(a_n)_{n \geq k}$ be a bounded sequence and let $a$ be a real number. Then $a_n \to a \iff \underline{\lim}(a_n) = \overline{\lim}(a_n) = a$.*

## Proof

*($\Rightarrow$):*
*Assume $a_n \to a$. Then all subsequences $(a_{n_k})$ also converge to $a$ and so $S = \{a\}$. Thus, $\inf S = \sup S = a$. Thus, by definition, $\underline{\lim}(a_n) = \overline{\lim}(a_n)$.*
*($\Leftarrow$):*
*We assume $\underline{\lim}(a_n) = \overline{\lim}(a_n)$ and so we have $S = \{a\}$. Suppose $a_n \not\to a$. Then there is an $\epsilon_0$ so that for all $k$, there is an $n_k$ with $|a_{n_k} - a| \geq \epsilon_0$. Since $(a_n)$ is bounded, $(a_{n_k})$ is also bounded. By the Bolzano Weierstrass Theorem, there is a subsequence $(a_{n_{k_p}})$ which we will denote by $(a^1_{n_k})$. We will let $n_{k_p}$ be denoted by $n^1_k$ for convenience.*

**Proof**

*The superscript 1 plays the role of adding another level of subscripting which is pretty ugly! This subsequence of the subsequence converges to a number* $y$. *So by definition,* $y \in S$. *But* $S$ *is just one point,* $a$, *so we have* $y = a$ *and we have shown* $a^1_{n_k} \rightarrow a$ *too. Now pick the tolerance* $\epsilon_0$ *for this sub subsequence. Then there is an* $Q$ *so that* $|a^1_{n_k} - a| < \epsilon_0$ *when* $n^1_k > Q$. *But for an index* $n^1_k > Q$, *we have both* $|a^1_{n_k} - a| < \epsilon_0$ *and* $|a^1_{n_k} - a| \geq \epsilon_0$. *This is not possible. Hence, our assumption that* $a_n \not\rightarrow a$ *is wrong and we have* $a_n \rightarrow a$. □

**Example**

$(a_n) = (\sin(n\pi/6))_{n \geq 1}$ has $S = \{-1, -\sqrt{3}/2, -1/2, 0, 1/2, \sqrt{3}/2, 1\}$. So $\underline{\lim}(a_n) = -1$ and $\overline{\lim}(a_n) = 1$ which are not equal. This tells us immendiately, $\lim(a_n)$ does not exist.

We can approach the inferior and superior limit another way.

**Definition**

Let $(a_n)$ be a bounded sequence. Define sequences $(y_k)$ and $(z_k)$ by

$y_k = \inf\{a_k, a_{k+1}, a_{k+2}, \ldots\} = \inf_{n \geq k}(a_n)$ and

$z_k = \sup\{a_k, a_{k+1}, a_{k+2}, \ldots\} = \sup_{n \geq k}(a_n)$.

Then we have

$y_1 \leq y_2 \leq \ldots \leq y_k \leq \ldots \leq B$ and $z_1 \geq z_2 \geq \ldots \geq z_k \geq \ldots \geq -B$ where $B$ is the bound for the sequence.

We see $y = \sup(y_k) = \lim_{k \to \infty} y_k$ and $z = \inf(z_k) = \lim_{k \to \infty} z_k$.

We denote $z$ by $\overline{\lim}^*(a_n)$ and $y$ by $\underline{\lim}_*(a_n)$.

Since $y_k \leq z_k$ for all $k$, we also know $\lim_k y_k = y \leq \lim_k z_k = z$.

We will show $\underline{\lim}_*(a_n) = \underline{\lim}(a_n)$ and $\overline{\lim}^*(a_n) = \overline{\lim}(a_n)$. Thus, we have two ways to characterize the limit inferior and limit superior of a sequence. Sometimes one is easier to use than the other!

Let's look more closely at the connections between subsequential limits and the $\underline{\lim}_*(a_n)$ and $\overline{\lim}^*(a_n)$.

## Theorem

*There are subsequential limits that equal* $\overline{\lim}^*(a_n)$ *and* $\underline{\lim}_*(a_n)$.

## Proof

*Let's look at the case for* $z = \overline{\lim}_*(a_n)$. *Pick any* $\epsilon = 1/k$.
*Let* $S_k = \{a_k, a_{k+1}, \ldots\}$. *Since* $z_k = \sup S_k$, *applying the Supremum Tolerance Lemma to the set* $S_k$, *there are sequence values* $a_{n_k}$ *with* $n_k \geq k$ *so that* $z_k - 1/k < a_{n_k} \leq z_k$ *for all* $k$.

*Thus,* $-1/k < a_{n_k} - z_k \leq 0 < 1/k$ *or* $|a_{n_k} - z_k| < 1/k$.

*Pick an arbitrary* $\epsilon > 0$ *and choose* $N_1$ *so that* $1/N_1 < \epsilon/2$.
*Then,* $k > N_1 \Rightarrow |a_{n_k} - z_k| < \epsilon/2$.
*We also know since* $z_k \to z$ *that there is an* $N_2$ *so that* $k > N_2 \Rightarrow |z_k - z| < \epsilon/2$.

**Proof**

*Now pick* $k > \max\{N_1, N_2\}$ *and consider*

$$\begin{aligned} |a_{n_k} - z| &= |a_{n_k} - z_k + z_k - z| \leq |a_{n_k} - z_k| + |z_k - z| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

*This shows* $a_{n_k} \rightarrow z$.
*A very similar argument shows that we can find a subsequence* $(a'_{n_k})$ *which converges to* $y$. *These arguments shows us* $y$ *and* $z$ *are in* $S$, *the set of all subsequential limits.* □

### Theorem

$$
\begin{aligned}
y = \underline{\lim}_*(a_n) \quad &\Longleftrightarrow \quad (c < y \Rightarrow a_n < c \textit{ for only finitely many indices })\\
&\quad \textit{and} \quad (y < c \Rightarrow a_n < c \textit{ for infinitely many indices })\\
z = \overline{\lim}^*(a_n) \quad &\Longleftrightarrow \quad (c < z \Rightarrow a_n > c \textit{ for infinitely many indices })\\
&\quad \textit{and} \quad (z < c \Rightarrow a_n > c \textit{ for only finitely many indices })
\end{aligned}
$$

## Proof

*Let's look at the proposition for* $\underline{\lim}_*(a_n) = y$. *The proof of the other one is similar and is worth your time to figure out!*
*(*$\Rightarrow$*):*
*We assume* $y = \underline{\lim}_*(a_n)$.
*For* $c < y = \sup\{y_k\}$, *there has to be a* $y_{k_0} > c$.
*Now* $y_{k_0} = \inf\{a_{k_0}, a_{k_0+1}, \ldots\}$ *and so we have* $a_n \geq y_{k_0} > c$ *for all* $n \geq k_0$.

*Hence, the set of indices where the reverse inequality holds must be finite. That is,* $\{n : a_n < c\}$ *is a finite set. This shows the first part.*

*Next, assume* $y < c$. *Thus,* $y_k < c$ *for all* $k$. *Hence,* $y_k = \inf\{a_k, a_{k+1}, \ldots\} < c$. *Let* $\epsilon = c - \inf\{a_k, a_{k+1}, \ldots\} = c - y_k > 0$. *Note,* $y_k + \epsilon = c$.

*Now, by the Infimum Tolerance Lemma, there is* $a_{n_k}$ *so that* $y_k \leq a_{n_k} < y_k + \epsilon = c$. *But we can do this for each choice of* $k$. *This shows the set* $\{n : a_n < c\}$ *must be infinite.*

## Proof

*($\Leftarrow$):*
*We will just show the first piece. We must show $A = y$. Since $A$ satisfies both conditions, we have*

*if $c < A$, $a_n < c$ for only finitely many indices* **and** *if $A < c$, $a_n < c$ for infinitely many indices.*

*So given $c < A$, since $a_n < c$ for only finitely many indices, we can find an index $k_0$ so that $a_n \geq c$ for all $k \geq k_0$.*
*This tells us $y_{k_0} = \inf\{a_{k_0}, a_{k_0+1}, \ldots\} \geq c$ also. But then we have $y_k \geq y_{k_0} \geq c$ for all $k \geq k_0$ too. But this tells us $y = \sup y_k \geq c$.*
*Now $y \geq c < A$ for all such $c$ implies $y \geq A$ as well.*

## Proof

*Now assume $A < c$.*
*So by assumption $a_n < c$ for infinitely many indices. Then we have $y_k = \inf\{a_k, a_{k+1}, \ldots\} < c$ also for all $k$.*
*Since $y_k \rightarrow y$, we see $y \leq c$ too.*
*Then since this is true for all $A < c$, we have $y \leq A$ also.*

*Combining, we have $y = A$ as desired. The argument for the other case is very similar. We will leave that to you and you should try to work it out as it is part of your growth in this way of thinking!* □

## Theorem

$$\lim(a_n) = a \Longleftrightarrow \underline{\lim}_*(a_n) = \overline{\lim}^*(a_n) = a$$

## Proof

*(⇒):*

*We assume* $\lim(a_n) = a$. *We also have* $y = \underline{\lim}_*(a_n)$ *and* $z = \overline{\lim}^*(a_n)$. *Let's assume* $y < z$.

*Then pick arbitrary numbers* $c$ *and* $d$ *so that* $y < d < c < z$. *Now use the previous Theorem. We have* $a_n < d$ *for infinitely many indices and* $a_n > c$ *for infinitely many indices also.*

*The indices with* $a_n > c$ *define a subsequence,* $(a_{n_k}) \subseteq (a_n)$. *Since this subsequence is bounded below by* $c$ *and it is part of a bounded sequence, the Bolzano Weierstrass Theorem tells us this subsequence has a convergent subsequence.*

*Call this subsequence* $(a^1_{n_k})$ *and let* $a^1_{n_k} \to u$. *Then* $u \geq c$. *Further, since* $a_n \to a$, *we must have* $u = a \geq c$.

*We can do the same sort of argument with the indices where* $a_n < d$ *to find a subsequence* $(a^1_{m_k})$ *of* $(a_n)$ *which converges to a point* $v \leq d$. *But since* $a_n \to a$, *we must have* $v = a \leq d$.

## Proof

*This shows* $a \leq d < c \leq a$ *which is impossible as* $d < c$. *Thus, our assumption that* $y < z$ *is wrong and we must have* $y = z$.
*(*$\Leftarrow$*):*
*Now we assume* $y = z$.
*Using what we know about* $y$, *given* $\epsilon > 0$, $y - \epsilon/2 < y$ *and so* $a_n < y - \epsilon/2$ *for only a finite number of indices.*

*So there is an* $N_1$ *so that* $a_n \geq y - \epsilon/2$ *when* $n > N_1$.
**This used the** $y$ **part of the IFF characterization of** $y$ **and** $z$.

*But* $y = z$, *so* **we can also use the characterization of** $z$.

*Since* $z = y < y + \epsilon/2$, $a_n > y + \epsilon/2$ *for only a finite number of indices. Thus, there is an* $N_2$ *so that* $a_n \leq y + \epsilon/2$ *for all* $n > N_2$.

*We conclude if* $n > \max\{N_1, N_2\}$, *then* $y - \epsilon/2 \leq a_n \leq y + \epsilon/2$ *which implies* $|a_n - y| < \epsilon$. *We conclude* $a_n \to y$ *and so*
$\lim(a_n) = \underline{\lim}_*(a_n) = \overline{\lim}^*(a_n)$. □

## Theorem

*For the bounded sequence* $(a_n)$, $\underline{\lim}_*(a_n) = \underline{\lim}(a_n)$ *and* $\overline{\lim}^*(a_n) = \overline{\lim}(a_n)$.

## Proof

*Since we can find subsequences* $(a_{n_k})$ *and* $(a'_{n_k})$ *so that* $\underline{\lim}_*(a_n) = \lim a_{n_k}$ *and* $\overline{\lim}^*(a_n) = \lim a'_{n_k}$, *we know* $\underline{\lim}_*(a_n)$ *and* $\overline{\lim}^*(a_n)$ *are subsequential limits. Thus, by definition,* $\underline{\lim}(a_n) \leq \underline{\lim}_*(a_n) \leq \overline{\lim}^*(a_n) \leq \overline{\lim}(a_n)$.

*Now let* $c$ *be any subsequential limit. Then there is a subsequence* $(a_{n_k})$ *so that* $\lim_k a_{n_k} = c$. *Hence, we know* $\underline{\lim}_*(a_{n_k}) = \overline{\lim}^*(a_{n_k}) = c$ *also. We also know, from their definitions,* $\underline{\lim}_*(a_{n_k}) \geq \underline{\lim}_*(a_n)$ *and* $\overline{\lim}^*(a_{n_k}) \leq \overline{\lim}^*(a_n)$. *Thus,*

$$\underline{\lim}_*(a_n) \quad \leq \quad \underline{\lim}_*(a_{n_k}) = \overline{\lim}^*(a_{n_k}) = c \leq \overline{\lim}^*(a_n).$$

## Proof

*Now, since the subsequential limit value $c$ is arbitrary, we have $\underline{\lim}_*(a_n)$ is a lower bound of the set of subsequential limits, $S$, and so by definition $\underline{\lim}_*(a_n) \leq \underline{\lim}(a_n)$ as $\underline{\lim}(a_n) = \inf S$.*

*We also know $\overline{\lim}^*(a_n)$ is an upper bound for $S$ and so $\overline{\lim}(a_n) \leq \overline{\lim}^*(a_n)$.*

*Combining inequalities we have*
$\overline{\lim}(a_n) \leq \overline{\lim}^*(a_n) \leq \overline{\lim}(a_n)$ *and* $\underline{\lim}(a_n) \leq \underline{\lim}_*(a_n) \leq \underline{\lim}(a_n)$.
*This shows us* $\underline{\lim}_*(a_n) = \underline{\lim}(a_n)$ *and* $\overline{\lim}(a_n) = \overline{\lim}(a_n)$. ☐

## Example

Show if $c$ is a positive number, the $\lim_{n\to\infty}(c)^{1/n} = 1$.

## Solution

*First, look at* $c \geq 1$. *Then, we can say* $c = 1 + r$ *for some* $r \geq 0$. *Then* $c^{1/n} = (1+r)^{1/n} \geq 1$ *for all* $n$.

*Let* $y_n = (1+r)^{1/n}$. *Then*

$$c^{1/n} = y_n \text{ with } y_n \geq 1 \implies c^{1/n} - 1 = y_n - 1 \geq 0.$$

*Let* $x_n = y_n - 1 \geq 0$. *Then we have* $c^{1/n} = 1 + x_n$ *with* $x_n \geq 0$. *Using a POMI argument, we can show* $c = (1+x_n)^n \geq 1 + nx_n$. *Thus,* $0 \leq (c)^{1/n} - 1 = x_n \leq (c-1)/n$. *This show* $(c)^{1/n} \to 1$.

*If* $0 < c < 1$, $1/c \geq 1$ *and so if we rewrite this just right we can use our first argument. We have* $\lim_{n\to\infty}(c)^{1/n} = \lim_{n\to\infty} \frac{1}{(1/c)^{1/n}} = 1/1 = 1$.

# Homework 11

11.1 If $(a_n) \to a$ and $(b_n) \to b$ and we know $a_n \leq b_n$ for all $n$, prove $a \leq b$. This might seem hard, but pick $\epsilon > 0$ and write down the $\epsilon - N$ inequalities without using the absolute values. You should be able to see what to do from there.

11.2 Prove there is a subsequence $(a'_{n_k})$ which converges to $\underline{\lim}_*(a_n)$. This is like the one we did, but uses the Infimum Tolerance Lemma.

11.3 If $y \geq c$ for all $c < A$, then $y \geq A$ as well. The way to attack this is to look at the sequence $c_n = A - 1/n$. You should see what to do from there.