Exponential Functions and Taylor Series

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Outline

Theorem

$\lim_{k \to \infty}(x^k/k!) = 0$ *for all* x.

Proof

There is a k_0 *with* $|x|/k_0 < 1$. *Thus,*

$$\frac{|x|^{k_0+1}}{(k_0+1)!} = \frac{|x|}{k_0+1}\,\frac{|x|^{k_0}}{k_0!} \leq \frac{|x|}{k_0}\,\frac{|x|^{k_0}}{k_0!}$$

$$\frac{|x|^{k_0+2}}{(k_0+2)!} = \frac{|x|}{k_0+2}\,\frac{|x|}{k_0+1}\,\frac{|x|^{k_0}}{k_0!} < \left(\frac{|x|}{k_0}\right)^2 \frac{|x|^{k_0}}{k_0!}$$

$$\vdots$$

$$\frac{|x|^{k_0+j}}{(k_0+j)!} \leq \left(\frac{|x|}{k_0}\right)^j \frac{|x|^{k_0}}{k_0!}$$

Since $\frac{|x|}{k_0} < 1$, *this shows* $\lim_{k \to \infty}(x^k/k!) \to 0$ *as* $k \to \infty$. □

Although we do not have the best tools to analyze the function

$$S(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

let's see how far we can get. We start with this Theorem:

Theorem

$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^n}{k!}$ *converges for all* x.

Proof

Let $f(x) = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^k}{k!}$ *and* $f_n(x) = \sum_{k=0}^{n}\frac{x^k}{k!}$. *We need to show these limits exist and are finite.*

We will follow the argument we used in the earlier Theorem's proof with some changes.

Proof

$$
\begin{aligned}
\sum_{n=k+1}^{N} \frac{|x|^k}{k!} &= \frac{|x|^k}{(k+1)!} + \frac{|x|}{(k+2)!} + \ldots + \frac{|x|^{N-k}}{(k+(N-k)\,)!} \\
&= \frac{|x|^k}{k!}\left\{\frac{|x|}{(k+1)} + \ldots \frac{|x|^{N-k}}{(k+1)\ldots(k+(N-k)\,)}\right\} \\
&\leq \frac{|x|^k}{k!}\left\{\frac{|x|}{(k+1)} + \ldots \frac{|x|^{N-k}}{(k+1)^{N-k})}\right\} \leq \frac{|x|^k}{k!}\sum_{k=0}^{N-k} r^k
\end{aligned}
$$

where $r = \frac{|x|}{(k+1)}$. *We can then use the same expansion of powers of* r *as before to find*

$$
\sum_{n=k+1}^{N} \frac{|x|^k}{k!} \leq \frac{|x|^k}{k!}\left\{\frac{1-\Big(|x|/(k+1)\Big)^{N-k+1}}{1-|x|/(k+1)}\right\}
$$

Proof

Since there is a K_1 *so* $|x|/(k+1) < 1/2$ *for* $k > K_1$*, for all* $k > K_1$*,* $1/(1 - |x|/(k+1)) < 2$ *and so*

$$\lim_{N\to\infty} \left\{ \sum_{n=k+1}^{N} \frac{|x|^k}{k!} \right\} \leq \frac{|x|^k}{k!} \lim_{N\to\infty} 2(1 - (1/2)^{N-k+1}) = 2\frac{|x|^k}{k!}$$

Now pick any $k > K_1$*, then we see* $f(x)$*, the limit of an increasing sequence of terms, is bounded above which implies it converges to its supremum.*

$$\begin{aligned} f(|x|) &= 1 + |x| + |x|/2! + \ldots + |x|^k/k! + \lim_{N\to\infty} \sum_{n=k+1}^{N} \frac{|x|^k}{k!} \\ &\leq 1 + |x| + |x|/2! + \ldots + |x|^k/k! + 2\frac{|x|^k}{k!}. \end{aligned}$$

Proof

So we know $f(|x|)$ *has a limit for all* x. *To see* $f(x)$ *converges also, pick an arbitrary* $\epsilon > 0$ *and note since* $\frac{|x|^k}{k!} \to 0$ *as* $k \to \infty$, $\exists K \ni k > K$ *implies* $\frac{|x|^k}{k!} < \epsilon/2$.

So for $k > K$, *we have*

$$\lim_{N\to\infty} \sum_{n=k+1}^{N} \frac{|x|^k}{k!} \quad = \quad 2\,\frac{|x|^k}{k!} < \epsilon$$

Then, for $k > K$,

$$\left| f(x) - \sum_{n=0}^{k} \frac{x^k}{k!} \right| \quad \leq \quad \left| \lim_{N\to\infty} \sum_{n=k+1}^{N} \frac{x^k}{k!} \right| \leq \lim_{N\to\infty} \sum_{n=k+1}^{N} \frac{|x|^k}{k!} < \epsilon.$$

This says $\lim_{k\to\infty} \sum_{n=0}^{k} \frac{x^k}{k!} = f(x)$. □

Next, let's look at $S_n(x) = (1 + x/n)^n$ and $S(x) = \lim_{n\to\infty} S_n(x)$.

$$\begin{aligned} S_n(x) &= 1 + \sum_{k=1}^{n} \binom{n}{k} \frac{x^k}{n^k} = 1 + \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} \frac{x^k}{n^k} \\ &= 1 + \sum_{k=1}^{n} \frac{(n(n-1)(n-2)\ldots(n-(k-1))}{k!} \frac{x^k}{n^k} \\ &= 1 + \sum_{k=1}^{n} (1-1/n)\ldots(1-(k-1)/n) \frac{x^k}{k!} \end{aligned}$$

Consider

$$\left| \sum_{k=1}^{n} (1-1/n)\ldots(1-(k-1)/n) \frac{x^k}{k!} - \sum_{k=1}^{n} \frac{x^k}{k!} \right| =$$

$$\left| \sum_{k=1}^{n} ((1-1/n)\ldots(1-(k-1)/n) - 1) \frac{x^k}{k!} \right| < \left| \sum_{k=1}^{n} ((1+1/n) - 1) \frac{x^k}{k!} \right|$$

Then we have

$$\left| \sum_{k=1}^{n} (1 - 1/n) \ldots (1 - (k-1)/n) \ \frac{x^k}{k!} - \sum_{k=1}^{n} \frac{x^k}{k!} \right|$$

$$< \left| \sum_{k=1}^{n} \frac{x^k}{n\,k!} \right| \leq (1/n) \sum_{k=1}^{n} \frac{|x|^k}{k!} = \frac{f(|x|)}{n} \to 0 \text{ as } n \to \infty.$$

as since $f(|x|)$ is finite, the ratio $\frac{f(|x|)}{n}$ goes to zero as $n \to \infty$. So we know

Theorem

$$S(x) = \lim_{n \to \infty} S_n(x) \quad = \quad 1 + \lim_{n \to \infty} \sum_{k=1}^{n} \frac{x^k}{k!} = f(x).$$

Proof

We just did this argument. □

We usually abuse notation for these sorts of limits and we will start doing that now. The notation $\sum_{k=0}^{\infty} \frac{x^k}{k!} \equiv \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!}$.

Also, since we are always letting n go to ∞, we will start using $\lim_n$ to denote $\lim_{n\to\infty}$ for convenience.

Theorem

$S(x)$ is continuous for all x

Proof

Fix x_0 and consider any $x \in B_1(x_0)$. Note $S_n'(x) = n(1 + x/n)^{n-1}\,(1/n) = (1 + x/n)^{n-1}$. We see $S_n'(x) = S_n(x)\,(1 + x/n)^{-1}$. For each n, apply the MVT to find c_x^n between x_0 and x so that $\frac{|S_n(x) - S_n(x_0)|}{|x - x_0|} = |S_n'(c_x^n)|$.

Proof

Now $\lim_n(1+c_x^n/n)=1$, *so for n sufficiently large,* $1<|1+c_x^n/n|^{-1}<2$. *Hence, if n sufficiently large*

$$|(1+c_x^n/n)^n|<|S_n'(c_x^n)| \quad < \quad 2\,|(1+c_x^n/n)^n|$$

Also, if you look at $S_n(x)$ *for* $x_0-1<x<x_0+1$,

$$1+\frac{x_0-1}{n} \quad < \quad 1+\frac{x}{n}<1+\frac{x_0+1}{n}$$

implying $S_n(x_0-1)<S_n(x)<S_n(x_0+1)$. *Since these numbers could be negative, we see* $|S_n(c_x^n|\leq D(x_0)=\max\{|S_n(x_0-1)|,|S_n(x_0-1)|\}$ *which we can choose to be positive. Thus, since* $x_0-1<c_x^n<x_0+1$, $|S_n'(c_x^n)|<2D(x_0)$ *for n sufficiently large. This tells us* $\lim_n \frac{|S_n(x)-S_n(x_0)|}{|x-x_0|}\leq 2D(x_0)$. *Letting* $n\to\infty$, *we find*

$$\frac{|S(x)-S(x_0)|}{|x-x_0|} \quad \leq \quad 2D(x_0).$$

Proof

This shows $S(x)$ is continuous at x_0 as if $\epsilon > 0$ is chosen, if $|x - x_0| < \epsilon / D(x_0)$ we have $|S(x) - S(x_0)| < \epsilon$. □

Now pick any x that is irrational. Then let (x_p) be any sequence of rational numbers which converges to x. Then since
$S(x) = \lim_{p \to \infty} S(x_p)$, we have
$S(x) = \lim_{p \to \infty} (1 + x_p/n)^n = \lim_{p \to \infty} e^{x_p}$.

Hence, we can define $e^x = \lim_{p \to \infty} e^{x_p}$ and we have extended the definition of e^x off of $\mathbb{Q}$ to $\mathbb{R}$.

We conclude

(1) $e^x = \lim_n (1 + x/n)^n = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
(2) $e = \lim_n (1 + 1/n)^n = \sum_{k=0}^{\infty} \frac{1}{k!}$
(3) e^x is a continuous function of x.

Theorem

$(e^x)' = e^x$

Proof

Let (x_p) be any sequence converging to x_0. Then, we have

$$\frac{S_n(x_p) - S_n(x_0)}{x_p - x_0} = S_n'(c_x^{n,p}) = S_n(c_x^{n,p})\,(1 + c_x^{n,p}/n)^{-1}.$$

where $c_x^{n,p}$ is between x_0 and x_p implying $\lim_p c_x^{n,p} = x_0$.

Given $\epsilon > 0$ arbitrary, $\exists P \ni p > P \Rightarrow x_0 - \epsilon < c_x^{n,p} < x_0 + \epsilon$. We then have

$$1 + \frac{x_0 - \epsilon}{n} < 1 + \frac{c_x^{n,p}}{n} < 1 + \frac{x_0 + \epsilon}{n} \Longrightarrow$$
$$\left(1 + \frac{x_0 - \epsilon}{n}\right)^n < \left(1 + \frac{c_x^{n,p}}{n}\right)^n < \left(1 + \frac{x_0 + \epsilon}{n}\right)^n$$

Proof

This also tells us

$$\frac{1}{1+\frac{x_0+\epsilon}{n}} < \frac{1}{1+\frac{c_x^{n,p}}{n}} < \frac{1}{1+\frac{x_0-\epsilon}{n}}$$

and so

$$\begin{aligned}\left(1+\frac{x_0-\epsilon}{n}\right)^n \frac{1}{1+\frac{x_0+\epsilon}{n}} &< \left(1+\frac{c_x^{n,p}}{n}\right)^n \frac{1}{1+\frac{c_x^{n,p}}{n}} \\ &< \left(1+\frac{x_0+\epsilon}{n}\right)^n \frac{1}{1+\frac{1}{x_0-\epsilon}n}\end{aligned}$$

Now let $p \to \infty$ and $n \to \infty$ and find

$$e^{x_0-\epsilon} \cdot 1 < \lim_{p\to\infty} \lim_{n\to\infty} \left(1+\frac{c_x^{n,p}}{n}\right)^n \frac{1}{1+\frac{c_x^{n,p}}{n}} < e^{x_0+\epsilon} \cdot 1$$

Proof

Now, letting $\epsilon \to 0^+$, *we find*

$$\begin{aligned} e^{x_0} = \lim_{\epsilon \to 0^+} e^{x_0 - \epsilon} &\leq \lim_p \lim_n \left(1 + \frac{c_x^{n,p}}{n}\right)^n \frac{1}{1 + \frac{c_x^{n,p}}{n}} \\ &= \lim_p \lim_n \frac{S_n(x_p) - S_n(x_0)}{x_p - x_0} \leq \lim_{\epsilon \to 0^+} e^{x_0 + \epsilon} = e^{x_0} \end{aligned}$$

as e^x *is a continuous function. We conclude*

$$\lim_p \lim_n S_n(c_x^{n,p}) = \lim_p \left\{ \frac{S(x_p) - S(x_0)}{x_p - x_0} \right\} = e^{x_0}.$$

Since we can do this for any such sequence (x_n) *we have shown* e^x *is differentiable with* $(e^x)' = e^x$. $\square$

Theorem

(1) $e^{x+y} = e^x e^y$
(2) $e^{-x} = 1/e^x$
(3) $e^x e^{-y} = e^{x-y} = e^x/e^y$

Proof

(1):

$$\begin{aligned} e^x e^y &= \lim_n \left((1+x/n)^n \right) \lim_n \left((1+y/n)^n \right) \\ &= \lim_n \left((1+x/n)\ (1+y/n) \right)^n \\ &= \lim_n \left(1 + (x+y)/n + xy/n^2 \right)^n \\ &= \lim_n \left(1 + (x+y)/n + (xy/n)(1/n) \right)^n . \end{aligned}$$

Pick an $\epsilon > 0$. *Then*
$\exists N_1 \ni n > N_1 \Rightarrow |xy|/n < \epsilon$ *and* $\exists N_2 \ni n > N_2 \Rightarrow 1 - (x+y)/n > 0$.

Proof

So if $n > \max\{N_1, N_2\}$, *we have*

$$\begin{aligned} 1 + (x+y)/n - \epsilon &< 1 + (x+y)/n + xy/n^2 \\ &< 1 + (x+y)/n + \epsilon/n \\ \left(1 + (x+y)/n - \epsilon/n\right)^n &< \left(1 + (x+y)/n + xy/n^2\right)^n \\ &< \left(1 + (x+y)/n + \epsilon/n\right)^n \end{aligned}$$

because all terms are positive once n *is large enough. Now let* $n \to \infty$ *to find*

$$e^{x+y-\epsilon} \leq e^x e^y \leq e^{x+y+\epsilon}$$

Finally, let $\epsilon \to 0^+$ *to find*

$$e^{x+y} \leq e^x e^y \leq e^{x+y} \Rightarrow e^x e^y = e^{x+y}$$

which tells us (1) is true.

Proof

(2):

$$\begin{aligned} e^{-x}e^{x} &= \lim_n \left((1 - x/n)^n \right) \lim_n \left((1 + x/n)^n\right) \\ &= \lim_n \left((1 + x/n)(1 - x/n)\right)^n = \lim_n \left(1 - x^2/n^2\right)^n \\ &= \lim_n \left(1 - (x^2/n)(1/n)\right)^n \end{aligned}$$

Now argue just like in (1). Given $\epsilon > 0$, $\exists N \ni n > N \Rightarrow x^2/n < \epsilon$. *Thus,*

$$\begin{aligned} 1 - \epsilon/n &< 1 - x^2/n^2 < 1 + \epsilon/n \Rightarrow \\ \left(1 - \epsilon/n\right)^n &< \left(1 - x^2/n^2\right)^n < \left(1 + \epsilon/n\right)^n \Rightarrow \\ \lim_n \left(1 - \epsilon/n\right)^n &\leq \lim_n \left(1 - x^2/n^2\right)^n \leq \lim_n \left(1 + \epsilon/n\right)^n \end{aligned}$$

This says $e^{-\epsilon} \leq e^{-x}e^{x} \leq e^{\epsilon}$ *and as* $\epsilon \to 0$, *we obtain (2).*
(3): $e^x e^{-y} = e^{x+(-y)} = e^{x-y}$ *and* $e^x e^{-y} = e^x/e^y$ *and so (3) is true.* □

Let's look at Taylor polynomials for some familiar functions.

(1):
$\cos(x)$ has the following Taylor polynomials and error terms at the base point 0:

$$
\begin{aligned}
\cos(x) &= 1 + (\cos(x))^{(1)}(c_0)x, c_0 \text{ between } 0 \text{ and } x \\
\cos(x) &= 1 + 0x + (\cos(x))^{(2)}(c_1)x^2/2, c_1 \text{ between } 0 \text{ and } x \\
\cos(x) &= 1 + 0x - x^2/2 + (\cos(x))^{(3)}(c_2)x^3/3!, c_2 \text{ between } 0 \text{ and } x \\
\cos(x) &= 1 + 0x - x^2/2 + 0x^3/3! + (\cos(x))^{(4)}(c_3)x^4/4!, \\
& \quad c_3 \text{ between } 0 \text{ and } x \\
\cos(x) &= 1 + 0x - x^2/2 + 0x^3/3! + x^4/4! + (\cos(x))^{(5)}(c_4)x^5/5!, \\
& \quad c_3 \text{ between } 0 \text{ and } x \\
\ldots \\
\cos(x) &= \sum_{k=0}^{n} (-1)^k \, x^{2k}/(2k)! + (\cos(x))^{(2n+1)}(c_{2n+1})x^{2n+1}/(2n+1)!, \\
& \quad c_{2n+1} \text{ between } 0 \text{ and } x
\end{aligned}
$$

Since the $(2n)^{th}$ Taylor Polynomial of $\cos(x)$ at 0 is $p_{2n}(x,0) = \sum_{k=0}^{n} (-1)^k \, x^{2k}/(2k)!$, this tells us that

$$\cos(x) - p_{2n}(x,0) \quad = \quad (\cos(x))^{(2n+1)}(c_{2n+1})x^{2n+1}/(2n+1)!.$$

The biggest the derivatives of $\cos(x)$ can be here in absolute value is 1. Thus $|\cos(x) - p_{2n}(x,0)| \leq (1)|x|^{2n+1}/(2n+1)!$ and we know

$$\lim_n |\cos(x) - p_{2n}(x,0)| \quad = \quad \lim_n |x|^{2n+1}/(2n+1)! = 0$$

Thus, the Taylor polynomials of $\cos(x)$ converge to $\cos(x)$ at each x. We would say $\cos(x) = \sum_{k=0}^{\infty} (-1)^k \, x^{2k}/(2k)!$ and call this the Taylor Series of $\cos(x)$ at 0

We can do this for functions with easy derivatives.

So we can show at $x = 0$
(1) $\sin(x) = \sum_{k=0}^{\infty}(-1)^k\, x^{2k+1}/(2k+1)!$
(2) $\sinh(x) = \sum_{k=0}^{\infty} x^{2k+1}/(2k+1)!$
(3) $\cosh(x) = \sum_{k=0}^{\infty} x^{2k}/(2k)!$
(4) $e^x = f(x) = \sum_{k=0}^{\infty} x^k/k!$

It is very hard to do this for functions whose higher order derivatives require product and quotient rules. So what we know that we can do in theory is not necessarily what we can do in practice.

Homework 28

28.1 Prove the n^{th} order Taylor polynomial of e^x at $x = 0$ is $f_n(x) = \sum_{k=0}^{n} x^k/k!$ and show the error term goes to zero as $n \to \infty$.

28.2 Find the n^{th} order Taylor polynomial of $\sin(x)$ at $x = 0$ and show the error term goes to zero as $n \to \infty$.

28.3 Find the n^{th} order Taylor polynomial of $\cosh(x)$ at $x = 0$ and show the error term goes to zero as $n \to \infty$.