Upper and Lower Bounds

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Outline

1. Upper and Lower Bounds
2. Examples
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4. Homework
Let $S$ be a set of real numbers. We need to make precise the idea of a set of real numbers being **bounded**.

**Definition**

We say a set $S$ is bounded above if there is a number $M$ so that $x \leq M$ for all $x$ in $S$. We call $M$ an **upper bound** of $S$ or just an u.b.

**Example**

If $S = \{ y : y = x^2 \text{ and } -1 \leq x \leq 2 \}$, there are many u.b.’s of $S$. Some choices are $M = 5$, $M = 4.1$. Note $M = 1.9$ is not an u.b.

**Example**

If $S = \{ y : y = \tanh(x) \text{ and } x \in \mathbb{R} \}$, there are many u.b.’s of $S$. Some choices are $M = 2$, $M = 2.1$. Note $M = 0$ is not an u.b. Draw a picture of this graph too.
Let $S$ be a set of real numbers.

**Definition**

We say a set $S$ is bounded below if there is a number $m$ so that $x \geq m$ for all $x$ in $S$. We call $m$ a **lower bound** of $S$ or just a **l.b.**

**Example**

If $S = \{y : y = x^2 \text{ and } -1 \leq x \leq 2\}$, there are many l.b.’s of $S$. Some choices are $m = -2$, $m = -0.1$. Note $m = 0.3$ is **not** a l.b.

**Example**

If $S = \{y : y = \tanh(x) \text{ and } x \in \mathbb{R}\}$, there are many l.b.’s of $S$. Some choices are $m = -1.1$, $m = -1.05$. Note $m = -0.87$ is **not** a l.b. Draw a picture of this graph again.
Let $S$ be a set of real numbers.

**Definition**

We say a set $S$ is bounded if $S$ is bounded above and bounded below. That is, there are finite numbers $m$ and $M$ so that $m \leq x \leq M$ for all $x \in S$. We usually overestimate the bound even more and say $B$ is bounded if we can find a number $B$ so that $|x| \leq B$ for all $x \in S$. A good choice of such a $B$ is to let $B = \max(|m|, |M|)$ for any choice of l.b. $m$ and u.b. $M$.

**Example**

If $S = \{y : y = x^2$ and $-1 \leq x < 2\}$. here $S = [0, 4)$ and so for $m = -2$ and $M = 5$, a choice of $B$ is $B = 5$. Of course, there are many other choices of $B$.

**Example**

If $S = \{y : y = \tanh(x)$ and $x \in \mathbb{R}\}$, we have $S = (-1, 1)$ and for $m = -1.1$ and $M = 1.2$, a choice of $B$ is $B = 1.2$. 
The next material is more abstract! We need to introduce the notion of **least upper bound** and **greatest lower bound**.
We also call the **least upper bound** the **l.u.b.**. It is also called the **supremum** of the set $S$. We use the notation $\sup(S)$ as well.
We also call the **greatest lower bound** the **g.l.b.**. It is also called the **infimum** of the set $S$. We use the notation $\inf(S)$ as well.

**Definition**

The **least upper bound**, **l.u.b.** or sup of the set $S$ is a number $U$ satisfying

1. $U$ is an upper bound of $S$
2. If $M$ is any other upper bound of $S$, then $U \leq M$.

The **greatest lower bound**, **g.l.b.** or inf of the set $S$ is a number $u$ satisfying

1. $u$ is a lower bound of $S$
2. If $m$ is any other lower bound of $S$, then $u \geq m$. 
Example

If \( S = \{ y : y = x^2 \text{ and } -1 \leq x < 2 \} \). here \( S = [0, 4) \) and so \( \inf(S) = 0 \) and \( \sup(S) = 4 \).

Example

If \( S = \{ y : y = \tanh(x) \text{ and } x \in \mathbb{R} \} \), we have \( \inf(S) = -1 \) and \( \sup(S) = 1 \). Not the \( \inf \) and sup of a set \( S \) need \textbf{NOT} be in \( S \)!

Example

If \( S = \{ y : \cos(2n\pi/3), \ \forall n \in \mathbb{N} \} \), The only possible values in \( S \) are \( \cos(2\pi/3) = -1/2, \cos(4\pi/3) = -1/2 \) and \( \cos(6\pi/3) = 1 \). There are no other values and these 2 values are endlessly repeated in a cycle. Here \( \inf(S) = -1/2 \) and \( \sup(S) = 1 \).
Comment

If a set $S$ has no finite lower bound, we set $\inf(S) = -\infty$. If a set $S$ has no finite upper bound, we set $\sup(S) = \infty$.

Comment

If the set $S = \emptyset$, we set $\inf(S) = \infty$ and $\sup(S) = -\infty$.

Definition

We say $Q \in S$ is a maximum of $S$ if $\sup(S) = Q$. This is the same, of course, as saying $x \leq Q$ for all $x$ in $S$ which is the usual definition of an upper bound. But this is different as $Q$ is in $S$. We call $Q$ a maximizer or maximum element of $S$.

We say $q \in S$ is a minimum of $S$ if $\inf(S) = q$. Again, this is the same as saying $x \geq q$ for all $x$ in $S$ which is the usual definition of a lower bound. But this is different as $q$ is in $S$. We call $q$ a minimizer minimal element of $S$. 
There is a fundamental *axiom* about the behavior of the real numbers which is very important.

**Axiom**

**The Completeness Axiom**

Let $S$ be a set of real numbers which is nonempty and bounded above. Then the supremum of $S$ exists and is finite.

Let $S$ be a set of real numbers which is nonempty and bounded below. Then the infimum of $S$ exists and is finite.

**Comment**

*So nonempty bounded sets of real numbers always have a finite infimum and supremum. This does not say the set has a finite minimum and finite maximum. Another way of saying this is that we don’t know if $S$ has a minimizer and maximizer.*
Theorem

Let $S$ be a nonempty set of real numbers which is bounded above. Then $\sup(S)$ exists and is finite. Then $S$ has a maximal element if and only if (IFF) $\sup(S) \in S$.

Proof

$(\Leftarrow)$: Assume $\sup(S)$ is in $S$. By definition, $\sup(S)$ is an upper bound of $S$ and so must satisfy $x \leq \sup(S)$ for all $x$ in $S$. This says $\sup(S)$ is a maximizer of $S$.

$(\Rightarrow)$: Let $Q$ denote a maximizer of $S$. Then by definition $x \leq Q$ for all $x$ in $S$ and is an upper bound. So by the definition of a supremum, $\sup(S) \leq Q$. Since $Q$ is a maximizer, $Q$ is in $S$ and from the definition of upper bound, we have $Q \leq \sup(S)$ as well. This says $\sup(S) \leq Q \leq \sup(S)$ or $\sup(S) = Q$. □
Theorem

Let $S$ be a nonempty set of real numbers which is bounded below. Then $\inf(S)$ exists and is finite. Then $S$ has a minimal element $\iff \inf(S) \in S$.

Proof

$(\Leftarrow)$: Assume $\inf(S)$ is in $S$. By definition, $\inf(S)$ is a lower bound of $S$ and so must satisfy $x \geq \inf(S)$ for all $x$ in $S$. This says $\inf(S)$ is a minimizer of $S$.

$(\Rightarrow)$: Let $q$ denote a minimizer of $S$. Then by definition $x \geq q$ for all $x$ in $S$ and is a lower bound. So by the definition of an infimum, $q \leq \inf(S)$. Since $q$ is a minimizer, $q$ is in $S$ and from the definition of lower bound, we have $\inf(S) \leq q$ as well. This says $\inf(S) \leq q \leq \inf(S)$ or $\inf(S) = q$. □
**Lemma**

**Infimum Tolerance Lemma:** Let $S$ be a nonempty set of real numbers that is bounded below. Let $\epsilon > 0$ be arbitrarily chosen. Then

$$\exists \ y \in S \ \exists \ \inf(S) \leq y < \inf(S) + \epsilon$$

**Proof**

*We do this by contradiction. Assume this is not true for some $\epsilon > 0$. Then for all $y$ in $S$, we must have $y \geq \inf(S) + \epsilon$. But this says $\inf(S) + \epsilon$ must be a lower bound of $S$. So by the definition of infimum, we must have $\inf(S) \geq \inf(S) + \epsilon$ for a positive epsilon which is impossible. Thus our assumption is wrong and we must be able to find at least one $y$ in $S$ that satisfies $\inf(S) \leq y < \inf(S) + \epsilon$. ☑️*
Lemma

**Supremum Tolerance Lemma**: Let $S$ be a nonempty set of real numbers that is bounded above. Let $\epsilon > 0$ be arbitrarily chosen. Then

$$\exists \ y \in S \ \exists \ sup(S) - \epsilon < y \leq sup(S)$$

Proof

We do this by contradiction. Assume this is not true for some $\epsilon > 0$. Then for all $y$ in $S$, we must have $y \leq sup(S) - \epsilon$. But this says $sup(S) - \epsilon$ must be an upper bound of $S$. So by the definition of supremum, we must have $sup(S) \leq sup(S) - \epsilon$ for a positive epsilon which is impossible. Thus our assumption is wrong and we must be able to find at least one $y$ in $S$ that satisfies $sup(S) - \epsilon < y \leq sup(S)$. \qed
Example

Let \( f(x, y) = x + 2y \) and let \( S = [0, 1] \times [1, 3] \) which is also \( S_x \times S_y \) where \( S_x = \{ x : 0 \leq x \leq 1 \} \) and \( S_y = \{ y : 1 \leq y \leq 3 \} \). Note
\[
\inf_{(x, y) \in [0,1] \times [1,3]} f(x, y) = 0 + 2 = 2 \quad \text{and} \quad \sup_{(x, y) \in [0,1] \times [1,3]} f(x, y) = 1 + 6 = 7.
\]

\[
\begin{align*}
\inf_{1 \leq y \leq 3} f(x, y) &= \inf_{1 \leq y \leq 3} (x + 2y) = x + 2 \\
\sup_{0 \leq x \leq 1} f(x, y) &= \sup_{0 \leq x \leq 1} (x + 2y) = 1 + 2y \\
\sup_{0 \leq x \leq 1} \inf_{1 \leq y \leq 3} (x + 2y) &= \sup_{0 \leq x \leq 1} (x + 2) = 3 \\
\inf_{1 \leq y \leq 3} \sup_{0 \leq x \leq 1} (x + 2y) &= \inf_{1 \leq y \leq 3} (1 + 2y) = 3
\end{align*}
\]

so in this example,
\[
\inf_{y \in S_y} \sup_{x \in S_x} f(x, y) = \sup_{x \in S_x} \inf_{y \in S_y} f(x, y)
\]
Example

Please look the text where you can see this function drawn out. That might help. Let

\[ f(x, y) = \begin{cases} 
0, & (x, y) \in (1/2, 1] \times (1/2, 1] \\
2, & (x, y) \in (1/2, 1] \times [0, 1/2] \text{ and } [0, 1/2] \times (1/2, 1] \\
1, & (x, y) \in [0, 1/2] \times [0, 1/2] 
\end{cases} \]

and let \( S = [0, 1] \times [0, 1] \) which is also \( S_x \times S_y \) where \( S_x = \{ x : 0 \leq x \leq 1 \} \) and \( S_y = \{ y : 0 \leq y \leq 1 \} \). Note
\[ \inf_{(x,y)\in[0,1] \times [0,1]} f(x, y) = 0 \text{ and } \sup_{(x,y)\in[0,1] \times [0,1]} f(x, y) = 2. \]
Then, we also can find
\[ \inf_{0\leq y \leq 1} f(x, y) = \begin{cases} 
1, & 0 \leq x \leq 1/2 \\
0, & 1/2 < x \leq 1 
\end{cases} \]
and
\[ \sup_{0\leq x \leq 1} f(x, y) = \begin{cases} 
2, & 0 \leq y \leq 1/2 \\
2, & 1/2 < y \leq 1 
\end{cases} \]
Example (Continued)

\[ \sup_{0 \leq x \leq 1} \inf_{0 \leq y \leq 1} f(x, y) = \sup_{0 \leq x \leq 1} \begin{cases} 1, & 0 \leq x \leq 1/2 \\ 0, & 1/2 < x \leq 1 \end{cases} = 1. \]

and

\[ \inf_{0 \leq y \leq 1} \sup_{0 \leq x \leq 1} f(x, y) = \inf_{0 \leq y \leq 1} \begin{cases} 2, & 0 \leq y \leq 1/2 \\ 2, & 1/2 < y \leq 1 \end{cases} = 2 \]

so in this example

\[ \inf_{y \in S_y} \sup_{x \in S_x} f(x, y) \neq \sup_{x \in S_x} \inf_{y \in S_y} f(x, y) \]

and in fact

\[ \sup_{x \in S_x} \inf_{y \in S_y} f(x, y) < \inf_{y \in S_y} \sup_{x \in S_x} f(x, y) \]
The moral here is that order matters. For example, in an applied optimization problem, it is not always true that

\[
\min_x \max_y f(x, y) = \max_y \min_x f(x, y)
\]

where \( x \) and \( y \) come from some domain set \( S \).
The moral here is that **order** matters. For example, in an applied optimization problem, it is not always true that

$$\min_x \max_y f(x, y) = \max_y \min_x f(x, y)$$

where \(x\) and \(y\) come from some domain set \(S\).

So it is probably important to find out when the order does not matter because it might be easier to compute in one ordering than another.
Theorem

Let $S$ be a nonempty bounded set of real numbers. Then $\inf(S)$ and $\sup(S)$ are unique.

Proof

By the completeness axiom, since $S$ is bounded and nonempty, we know $\inf(S)$ and $\sup(S)$ are finite numbers. Let $u_2$ satisfy the definition of supremum also. Then, we know $u_2 \leq M$ for all upper bounds $M$ of $S$ and in particular since $\sup(S)$ is an upper bound too, we must have $u_2 \leq \sup(S)$. But since $\sup(S)$ is a supremum, by definition, we also know $\sup(S) \leq u_2$ as $u_2$ is an upper bound. Combining, we have $u_2 \leq \sup(S) \leq u_2$ which tells us $u_2 = \sup(S)$. A similar argument shows the $\inf(S)$ is also unique. $\square$
4.1 Let

\[ f(x, y) = \begin{cases} 
3, & (x, y) \in (1/2, 1] \times (1/2, 1] \\
-2, & (x, y) \in (1/2, 1] \times [0, 1/2] \text{ and } [0, 1/2] \times (1/2, 1] \\
4, & (x, y) \in [0, 1/2] \times [0, 1/2] 
\end{cases} \]

and let \( S = [0, 1] \times [0, 1] \) which is also \( S_x \times S_y \) where \( S_x = \{x : 0 \leq x \leq 1\} \) and \( S_y = \{y : 0 \leq y \leq 1\} \). Find

1. \( \inf_{(x,y) \in S} f(x,y) \), and \( \sup_{(x,y) \in S} f(x,y) \),
2. \( \inf_{y \in S_y} \sup_{x \in S_x} f(x,y) \) and \( \sup_{x \in S_x} \inf_{y \in S_y} f(x,y) \),
3. \( \inf_{x \in S_x} \sup_{y \in S_y} f(x,y) \) and \( \sup_{y \in S_y} \inf_{x \in S_x} f(x,y) \).

4.2 Let \( S = \{z : z = e^{-x^2-y^2} \text{ for } (x, y) \in \mathbb{R}^2\} \). Find \( \inf(S) \) and \( \sup(S) \).

Does the minimum and maximum of \( S \) exist and if so what are their values?